College Admission with Multidimensional Privileges: The Brazilian Affirmative Action Case*

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Abstract

In August 2012 the Brazilian federal government enacted a law mandating the implementation of affirmative action policies in public federal universities for candidates from racial minorities, low-income families and those coming from public high schools. We show that the method proposed by the government transforms the students' affirmative action status into a strategic choice, and may in fact create situations where high-achieving students who belong to those groups are not accepted to a university they desire while a low-achieving student who does not belong to those groups is accepted. Data from university admissions in more than 3,000 programs in 2013 show evidence consistent with this type of unfairness in more than 49% of those programs' assignments. We propose a choice function for the colleges that removes any gain from strategizing over the privileges claimed, is fair and, under reasonable assumptions on the type distribution of the population, fully satisfies the diversity objectives expressed by the law. We also propose an incentive-compatible mechanism that matches students and colleges with the use of the proposed choice function.

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1 Introduction

Following an increasing need for affirmative action for students of African descent and of low-income families in terms of access to public universities[1], in August 2012 the Brazilian congress enacted a law [2] establishing the implementation of a series of affirmative action policies throughout the federal higher education system which has an annual inflow of hundreds of thousands of students to its undergraduate programs.

The law established that 50% of the seats in each program offered in those institutions[3] should be used for the affirmative action policies. In order to claim what we will denote the **privilege** of having higher priority access to those seats, a student must complete three years of high school at a public institution. When assigning students to at least 50% of those seats, the university should also give higher priority to students who claim the privilege associated with belonging to a low-income family (and provide documented proof). Additionally, when assigning a number of seats in the same proportion as the aggregate number of blacks, browns, and natives (here referred to as "minorities") in the state in which the institution is, the university should give higher priority to students who claim the privilege associated with being a minority. We denote those as "**public HS privilege**," "**low-income privilege**," and "**minority privilege**."

In a state where minorities constitute 25% of the population, for example, a program with 80 seats will have 40 seats giving higher priority to students claiming public HS privilege. At least 20 of those should give a higher priority to those claiming low-income privilege, and 10 for those claiming minority privilege.

One key distinctive issue presented by the privileges proposed in the law is the fact that they are multidimensional. That is, students may belong to one or more of the groups specified. For instance, a low-income, white student from a public high school qualifies for the low income and public HS privileges, but not for the minority privilege.

In October of the same year, Brazil's Ministry of Education published an ordinance[4] specifying some details on the implementation of the affirmative action law as well as a suggested mechanism for choosing students while satisfying those policies. Starting in the student selection processes of 2013, those recommendations were widely adopted as the new selection criteria. The mechanism suggested consisted of partitioning the set of seats in each program into five subsets, one for each valid combination of privileges claimed. Students would then choose which subset to apply for, among those for which they are eligible. Those with the highest exam scores were accepted, up to the capacity of each partition.

In this paper we show that the method proposed by the government has some important deficiencies. First, it is **unfair** in the sense that it may reject *high-achieving* students who are the target of the affirmative action policies while accepting *low-achieving* students who

[1] For detailed information on the history of affirmative action in Brazil, see Moehlecke (2002).

[2] 2012, Decreto no. 7824 –

[3] In Brazil, like in the Turkish system studied in Balinski and Sönmez (1999), students apply directly to the university for a specific program whereas in other countries like the US where students simply apply to the university and, once accepted, must then choose the majors or programs to pursue.

[4] 2012, Portaria Normativa 18 –

do not have privilege priority status. For example, a low-income black student with a high exam grade may be rejected while a high-income white student with a low exam grade is accepted. This is not just a theoretical observation: an analysis of the cutoff grades in the 2013 admissions show evidence consistent with unfairness in the assignments in about 49% of the more than 3,000 programs available. Second, it gives an advantage to students who **strategize over the privileges that they claim**. A student who makes that choice based on good information over other students' choices and their exam grades can improve their chances of being accepted at their preferred programs. Third, it **may not satisfy the affirmative action objectives** (that is, the ratios established in the law) even when these are feasible.

We provide a new choice procedure to be used by the programs that follows the affirmative action objectives but eliminates the problems above. It does so by guaranteeing that no student could be worse off by claiming additional privileges. Moreover, we show that the choice function that is defined by that procedure can be combined with the cumulative offer algorithm to provide a strategy-proof mechanism for matching students to programs under the proposed policies.

This paper is related to the literature on affirmative action in college admissions and school choice mechanisms. Similarly to the model for minority reserves of Hafalir et al. (2013) and the soft-bound quotas for multiple types in Ehlers et al. (2014) and Bo (2013), the affirmative action policy that is evaluated in our problem consists of giving a higher priority access of some seats for certain types of students. Our formulations of the currently used mechanism and our proposal are based on the slot-specific priorities model of Kominers and Sönmez (2012), where the seats in each college may have different priority orderings over the students and, moreover, these orderings may include different contractual terms between the students and the colleges. By using that model, we show that the incentives for manipulating the privileges that are submitted can be traced back to the priority orderings that are used for the seats, and that the solution relies on an alternative formulation of those priority orderings.

It is important to note that it would be possible for us to formulate our problem (and solution) in terms of the matching with complex constraints presented in Westkamp (2013), but due to the simplicity of representation of the entire argument in terms of slots and contracts, we opted to follow the model of Kominers and Sönmez (2012).

The remainder of this paper is structured as follows. In section 2 we present the mechanism suggested by the Ministry of Education and currently used by the universities surveyed. In section 3 we introduce the matching with contracts model that we apply to the college admissions problem with multidimensional privileges. In section 4 we show that the currently used Brazil Reserves choice function induces a game in which strategically sophisticated students may obtain better outcomes by strategizing over which privileges to claim. We also show that the current mechanism does not satisfy a fairness condition and cannot guarantee the satisfaction of the affirmative action objectives when they are feasible. Section 5 provides empirical evidence on how the situations that lead to those problems were pervasive in the year 2013. In section 6, we introduce the Multidimensional Brazil Privileges choice function and we build upon the choice function defined to describe a mechanism – the Student

Optimal Stable Mechanism – that matches students to colleges using a centralized system, satisfies stability, is strategy-proof, and fair. All the proofs are given in the Appendix.

2 The Ministry of Education's Guidelines

For the most part, until 2010, college admissions in Brazil essentially worked in a decentralized way. Students first applied to each university for a single program (e.g., history at University of Brasilia or biology at Federal University of Minas Gerais). Then, by using a combination of scores in a national exam and sometimes exams particular to those programs, the universities ranked them and accepted the top applicants to each program up to the programs' capacities, putting the remaining ones on waiting lists. Among those accepted, typically some would not enroll because they had also been accepted by other universities and courses of their preference. The universities would then proceed to a second round, accepting students from the wait list following their ranking. Depending on the university this might be followed by third, fourth or more rounds.

The introduction of the affirmative action law has not changed the decentralized nature of the entire system itself, but the affirmative action law changed the choice rules the universities use in each step in an attempt to satisfy the affirmative action objectives. In 2010 some universities started using a new centralized online system, denoted SiSU. An increasing number of universities and students are now using it to recruit students and choose programs. Our analysis and proposals, however, can be applied to improve both decentralized and centralized systems. The rules used by the universities surveyed in this work are, essentially, strict implementations (or small variations) of the one suggested by Brazil's Ministry of Education. This rule confirms the set of students to be chosen from any set of applicants and will be denoted as the class of *Brazil Reserves Choice Functions* (or simply *Brazil Reserves*). It suggests that the seats for each program should be split into five subsets. Let r be the proportion of minorities in the population of the state in which the program is. For any program with capacity q, the five distinct subsets are:

- A set Q_{MI} with $\left\lceil \frac{q}{4} r \right\rceil$ seats which give priority to students who claim public HS, minority, and low-income privileges.
- A set Q_{mI} with $\left\lceil \frac{q}{4}(1-r) \right\rceil$ seats which give priority to students who claim public HS and low-income privileges only.
- A set Q_{Mi} with $\left\lceil \frac{q}{4} r \right\rceil$ seats which give priority to students who claim public HS and minority privileges only.
- A set Q_{mi} with $\left\lceil \frac{q}{4}(1-r) \right\rceil$ seats which give priority to students who claim public HS privilege only.
- A set Q_{-} with the remaining q^{-} seats.

Given the students who apply for each of these subsets, those better ranked on the entrance exam are accepted up to the capacity of the set. It is easy to see that if there are enough applicants for each of those sets, the affirmative action objectives, as described by the law, are satisfied. In case the number of students who apply for some of those sets is smaller than their capacity, those seats are filled following the priority structure below:

- If there are seats free in Q_{MI}, those are made available:
  - to students claiming low income and public HS privileges only, then
  - to students claiming minority and public HS privileges only, then
  - to students claiming public HS privileges only, then
  - to any student.
- If there are seats available in Q_{mI}, those are made available:
  - to students claiming low income, minority, and public HS privileges, then
  - to students claiming minority and public HS privileges only, then
  - to students claiming HS privilege only, then
  - to any student.
- If there are seats available in Q_{Mi}, those are made available:
  - to students claiming public HS privilege only, then
  - to students claiming low income, minority, and public HS privileges, then
  - to students claiming low income and public HS privileges only, then
  - to any student.
- If there are seats available in Q_{mi}, those are made available:
  - to students claiming minority and public HS privileges only, then
  - to students claiming low income, minority, and public HS privileges, then
  - to students claiming low income and public HS privileges only, then
  - to any student.

It is not specified, however, in which order those seats are filled following those priorities.[5]

[5] In section 4.1 we present two actual implementations in use by universities surveyed, clarifying the order in which those seats are filled. Although not explicitly stated in the government's document, we assume that universities do not give higher priority to students claiming some privileges for the open access seats (Q_{-}).

3 Model

There are finite sets $S = \{s_1, \ldots, s_n\}$ and $P = \{p_1, \ldots, p_m\}$ of students and programs. The set $S^P \subseteq S$ contains all students in S from public high schools, $S^m \subseteq S^P$ contains the racial minority students from public schools and $S^i \subseteq S^P$ contains the low-income students from public schools. Each program p has its own capacity q_p and minority privilege ratio r_p. Each student s has a vector of exam scores $z(s) = (z_{p_1}(s), \ldots, z_{p_m}(s))$ such that $z_p(s)$ indicates the score of student s in program p. There are no ties in the exam grades of each program, that is, $z_p(s) = z_p(s') \implies s = s'$. For any two students $s, s' \in S$ and program $p \in P$, $z_p(s) = z_p(s') \iff s = s'$. Each student s has a vector of available privileges she can claim, $t_s = (t_s^p, t_s^m, t_s^i)$ where t_s^p, t_s^m, t_s^i represents public HS, minority, and low-income privileges, respectively, and strict preferences $>^*$ over programs in P and remaining unmatched. Each element of t_s is binary, where 1 means that the student is eligible for the privilege and 0 that she is not eligible. For example, if a student is a low-income, non-minority student from a public high school then $t_s = (1, 0, 1)$. To reduce confusion, we will typically represent this vector by using lowercase and uppercase letters, so $(1, 0, 1)$ will be represented by (P, m, I), for example. In the Brazilian system, if a student claims public HS, minority or low-income privileges she is required to prove those classifications.[6] Therefore, some students may opt to not claim a privilege associated with a group she belongs to, but students who do not belong to a group (and therefore do not have any proof of belonging to it) are unable to claim that privilege.

For simplicity, we will make use of the *matching with contracts* (Hatfield and Milgrom, 2005) notation. A **contract** x, in this context, is a tuple (s, p, t), where $s \in S$, $p \in P$ and $t = (t^p, t^m, t^i)$. The vector t represents the set of privileges the student claims. The values t^p, t^m, t^i are binary and represent claiming public HS, minority, and low-income privileges, respectively. A contract (s, p, t) is **valid** if $t = (t^p, t^m, t^i) \leqslant t_s$. For a contract x; x_S, x_P and x_T represent the student, program and vector of privileges that student s claims in contract x, respectively. Let X be the set of all valid contracts. For ease of notation, for a set of contracts Y, Y_i is the subset of Y that contains only the contracts that include $i \in S \cup P$. Similarly, Y_t is the subset of Y that only contains the contract with the privilege vector t. Let $s(Y)$, moreover, be the set of students with contracts in Y, that is, $s(Y) = \{s \in S : \exists (s, p, t) \in Y\}$. An **allocation** is a set of contracts $X' \subset X$, such that for every $s \in S$ and every $p \in P$, $|X'_s| \leqslant 1$ and $|X'_p| \leqslant q_p$. Let χ be the set of all possible allocations.

The null contract, meaning that the student has no contract, is denoted by $\varnothing$. Students have complete preferences, $(\succeq_s)_{s \in S}$, over contracts that include them and the null contract, $X_s \cup \varnothing$. These preferences are derived from students' strict preferences, $(\succ_s^*)_{s \in S}$, over programs and being unmatched, in addition to the fact that they place no relevance on how

[6] Unlike the public HS and low-income privileges, in order to claim minority privileges a student only has to identify herself as black, brown or native. Therefore, in principle, it is possible for a white student to declare herself as black. This possibility, however, is ignored in this paper.

they are accepted to a program:[7]

$$\forall s \in S, \forall p, p' \in P \text{ and } t, t' \leqslant t_s : (s,p,t) \succ_s (s,p',t') \iff p \succ_s^* p'$$

Next, the choice function of program p, $C_p : 2^X \to 2^X$ is such that for $Y \subset X$, $C_p(Y) \subset Y_p$. The set $C_p(Y)$ has a cardinality of at most q_p and has at most one contract per student.[8]

The choice functions that we will present in this paper are all instances of choice functions using *slot-specific priorities*, described in Kominers and Sönmez (2012). Under slot-specific priorities, each seat in a program has its own priority ordering over contracts. Given a set of contracts, each seat "accepts" the top contract with respect to that seat's priority ordering, among those who have not yet been accepted. As shown by the authors, the set of contracts accepted may depend on the order in which those seats are filled, and therefore that order is also a parameter of the problem.

More specifically, under slot-specific priorities, each seat i in a program has a priority order Π^i over elements of X, and each program p has an order of precedence over its seats $\rhd^p$. The interpretation of $i \rhd^p i'$ is that, whenever possible, the program p fills the seat i before filling i'. That is, when it is the turn of one seat, it will be filled by the contract with the highest priority, among those available to be chosen. As we will show, this model is rich enough for us to represent both the current procedures being used and our proposed solution.

A mechanism is a strategy space Δ_s for each student s along with an outcome function $\psi : \prod_{s \in S} \Delta_s \to \chi$ that selects an allocation for each strategy vector $\prod_{s \in S} \delta_s \in \prod_{s \in S} \Delta_s$. Given a student s and a strategy $\delta_s \in \Delta_s$, let δ_{-s} denote the strategy of all students except student s.

3.1 Desired properties of choice functions and mechanisms

Below we define properties that we consider desirable for the choice functions used by programs as well as for a centralized mechanism that assigns students to programs. The first one is privilege monotonicity.

Definition 1. Given a set of contracts X, a choice function $C : 2^X \to 2^X$ is **privilege monotonic** if for any given set of contracts $Y \subset X$, and any student s with no contract in Y,

$$(s,p,t_s) \notin C_p(Y \cup \{(s,p,t_s)\}) \implies (s,p,t') \notin C_p(Y \cup \{(s,p,t')\}), \forall t' \leqslant t_s.$$

Privilege monotonicity suggests that when a student applies to a program, claiming an additional privilege should not decrease her chance of being accepted. With this property,

[7] To the best of our knowledge, there is no objective reason to believe that students have preferences over how they are accepted, since whether a student was accepted because of a privilege is not made public, and no benefit or assistance given by the university is associated therewith.

[8] The assumption we use about student preferences is one of the differences between this paper and the current matching with contracts literature, since our model incorporates indifferences among some contracts, in contrast to the usual assumption of strict preferences found in the literature to date. Due to the indifferences that students have between some contracts, we cannot derive choice functions for the students as defined in the many-to-one matching with contracts models. As a result, instead of choice functions for students, we use students' preferences.

we can state that for any school, students do not have to gather information and strategize their application processes with respect to those privileges — leveling the playing field for students.

Definition 2. Given a set of contracts X, a choice function $C : 2^X \to 2^X$ is **fair** if for any given subset $Y \subset X$, any program p and $x \in Y_p$,

$$x \notin C_p(Y) \implies \forall y \in C_p(Y), \text{ either } z_p(y_S) > z_p(x_S) \text{ or } x_T \ngeqslant y_T \geqslant (P, m, i).$$

Fairness of the choice function as we use here indicates that if a student is not chosen, those contracts that are chosen include students who either have higher test scores or are there due to the affirmative action policy.

The new law enacted in Brazil requires some structure on the sets chosen by programs, with respect to the groups to which the students belong. In other words, the ratios associated with public HS, low income and minorities should, whenever possible, be satisfied by the students chosen for each program. We formalize this in the definition below.

Definition 3. A choice function $C_p : 2^X \to 2^X$ **satisfies the affirmative action objectives** at program p if $\forall Y \subset X$:

$$\begin{aligned} |\{x &\in C_p(Y) : x_T \geqslant (P, m, i)\}| \geqslant \min\{\frac{q_p}{2}, |\{x \in Y : x_T \geqslant (P, m, i)\}|\}, \\ |\{x &\in C_p(Y) : x_T \geqslant (P, m, I)\}| \geqslant \min\{\frac{q_p}{4}, |\{x \in Y : x_T \geqslant (P, m, I)\}|\}, \\ \text{and } |\{x &\in C_p(Y) : x_T \geqslant (P, M, i)\}| \geqslant \min\{\frac{r_p q_p}{2}, |\{x \in Y : x_T \geqslant (P, M, i)\}|\}. \end{aligned}$$

That is, in order to satisfy the affirmative action objectives, a choice function must select a sufficient number of students from all groups of students that are subject to affirmative action, whenever it is possible.

The definitions above apply to choice functions. Below, we provide properties for mechanisms and allocations.

Definition 4. An allocation X' is **fair** if for any given pair of contracts $x, y \in X'$

$$y_P \succ_{x_S} x_P \implies \text{either } z_{y_P}(y_S) > z_{y_P}(x_S) \text{ or } x_T \ngeqslant y_T \geqslant (P, m, i).$$

That is, an allocation is fair if the reason why a student is not matched to a program she prefers is because every student matched to that program either has a higher exam score or is from a public HS and is claiming strictly more privileges.

Definition 5. An allocation X' is **stable** if

i. for all $s \in S$ and for all $p \in P$, $X'_s \succ_s \varnothing$, $C_p(X') = X'_p$; and

ii. $\nexists (p, s) \in P \times S$, and contract $x \in X \backslash X'$, such that $x \in C_p((X' \backslash X'_s) \cup \{x\}), x \succ_s X'_s$.

Definition 6. A mechanism ψ is **incentive-compatible** if

$$\nexists s \in S, \delta_{-s} \in \prod_{j \in S \setminus \{s\}} \Delta_j, \delta' \in \Delta_s, \text{ such that } \psi(\delta', \delta_{-s})_s \succ_s \psi((t_s, \succ_s), \delta_{-s})_s.$$

In other words, for any student that we consider, no matter what her true preferences are or which privileges are available to her, it would be in her best interest to reveal her true preferences and claim all the privileges that she is eligible for.

4 Current Mechanism Revisited

So far, we have introduced some desired properties that choice functions and mechanisms should satisfy. In this section, we first formally describe two of the choice functions which are implementations of the guidelines published by the Ministry of Education and currently used by two of the largest federal universities in Brazil. Next, we show some deficiencies of those choice functions and any stable mechanism that uses them.

4.1 Two examples of the Brazil Reserves

Since the specification given by the guidelines allows for different choice procedures, we can find variation on the universities' implementation of it. We describe two instances: the choice functions used by the Federal University of Minas Gerais (UFMG) and by the Federal University of Rio Grande do Sul (UFRGS).[9]

Implementations of the Brazil Reserves, as those used by the universities analyzed in this section, use the priority orderings defined by the law and described in section 2. That is, for any given program, the numbers of seats in Q_{mi}, Q_{Mi}, Q_{mI}, Q_{MI} and Q_{-} are determined by the current guidelines and for a given set of contracts X' the priority structure in every program p satisfies the following:

- Seats in Q_{MI}:$X'_{(P,M,I)} \succ_p^{mi} X'_{(P,m,I)} \succ_p^{mi} X'_{(P,M,i)} \succ_p^{mi} X'_{(P,m,i)} \succ_p^{mi} X'_{(p,m,i)}$
- Seats in Q_{mI}:$X'_{(P,m,I)} \succ_p^{Mi} X'_{(P,M,I)} \succ_p^{Mi} X'_{(P,M,i)} \succ_p^{Mi} X'_{(P,m,i)} \succ_p^{Mi} X'_{(p,m,i)}$
- Seats in Q_{Mi}:$X'_{(P,M,i)} \succ_p^{mI} X'_{(P,m,i)} \succ_p^{mI} X'_{(P,M,I)} \succ_p^{mI} X'_{(P,m,I)} \succ_p^{mI} X'_{(p,m,i)}$
- Seats in Q_{mi}:$X'_{(P,m,i)} \succ_p^{MI} X'_{(P,M,i)} \succ_p^{MI} X'_{(P,M,I)} \succ_p^{MI} X'_{(P,m,I)} \succ_p^{MI} X'_{(p,m,i)}$

The priorities among contracts with the same privilege vector are determined by the students' exam grades: $(s,p,t) \succ_p (s',p,t) \iff z_p(s) > z_p(s')$. There are two dimensions that are not explicitly defined by the law and that are, in fact, different in both the UFMG and UFRGS implementations: the order of precedence and the priority ordering that is used in Q_{-} (open access seats).

[9]See 2012b, Edital UFMG – and 2012a, Edital UFRGS –.

For any given set of contracts X', the choice function used by a program p at the UFMG, C^{UFMG}, fills seats using the following order of precedence: $Q_{mi} \rhd^p Q_{Mi} \rhd^p Q_{mI} \rhd^p Q_{MI} \rhd^p Q_-$. For the last group, Q_-, the following priority ordering is used:

$$X'_{(p,m,i)} \succ_p^- X'_{(P,M,I)} \succ_p^- X'_{(P,m,I)} \succ_p^- X'_{(P,M,i)} \succ_p^- X'_{(P,m,i)}$$

The choice function used by UFRGS, C^{UFRGS}, differs in two ways from C^{UFMG}: the order in which the sets of seats are filled and the priority ordering used in Q_-. More specifically, the choice function fills seats using the following order of precedence: $Q_- \rhd^p Q_{MI} \rhd^p Q_{mI} \rhd^p Q_{Mi} \rhd^p Q_{mi}$. For the seats in Q_-, the priority ordering is strict with regard to the exam grades. That is, for $s \neq s'$, $(s,p,t) \succ_p^- (s',p,t') \iff z_p(s) > z_p(s')$ and for all s, p:

$$(s,p,(P,M,I)) \succ_p^- (s,p,(P,m,I)) \succ_p^- (s,p,(P,M,i)) \succ_p^- (s,p,(P,m,i)) \succ_p^- (s,p,(p,m,i))$$

4.2 The case against the current mechanism

Using examples, we now show that implementations of the Brazil Reserves choice function fail to satisfy the desirable properties defined in section 3.1. We will consider the case where $z_p(x_S^i) > z_p(x_S^j) \iff i < j$, that is, students' contract numbers are ordered by decreasing exam grades. We start with privilege monotonicity.

Example 1 (Privilege Monotonicity)**.** For a given program p, let $q_p = 8$, $r_p = \frac{1}{2}$ and let the set of contracts be $Y = \{x^1, \ldots, x^8\}$, where $x_T^1 = x_T^2 = x_T^3 = x_T^4 = (p,m,i)$, $x_T^5 = (P,m,i)$, $x_T^6 = (P,M,I)$, $x_T^7 = (P,M,i)$ and $x_T^8 = (P,m,I)$. Consider a low-income, minority student from public high school s, where $s \notin s(Y)$ and $z(x_S^6) > z_p(s) > z(x_S^8)$. If she applies with a contract that includes all of her privileges, i.e., $(s,p,(P,M,I))$, no matter which example of the Brazil Reserves the program p uses, she will be rejected:

$$(s,p,(P,M,I)) \notin C_p(Y \cup \{(s,p,(P,M,I))\}) = \{x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$$

However, if she claims only low-income and public HS privileges, i.e. $(s,p,(P,m,I))$, no matter which implementation of the BRCF the program p uses, her contract will be accepted:

$$(s,p,(P,m,I)) \in C_p(Y \cup \{(s,p,(P,m,I))\}) = \{x^1, x^2, x^3, x^4, x^5, x^6, x^7, (s,p,(P,m,I))\}$$

Therefore, implementations of the Brazil Reserves may not be privilege monotonic.

The example above shows that since the choice function gives the top priority at some seats to students who claim low income and public HS only, the choice function may incentivize a student to not claim her minority privilege. The second example we give regards the fairness property of choice functions.

Example 2 (Fairness of the Choice Function)**.** For a given program p, let $q_p = 8$, $r_p = \frac{1}{2}$ and let the set of contracts be $Y = \{x^1, \ldots, x^9\}$, where $x_T^1 = x_T^2 = x_T^3 = x_T^4 = (p,m,i)$,

$x_T^5 = x_T^6 = (P, M, I)$, $x_T^7 = (P, m, I)$, $x_T^8 = (P, M, i)$ and $x_T^9 = (P, m, i)$. In this case, no matter which example of the Brazil Reserves program p uses, the chosen set will be:

$$C_p(Y) = \{x^1, x^2, x^3, x^4, x^5, x^7, x^8, x^9\}$$

since $z(x^9) < z(x^6)$ and $x_t^6 \geqslant x_t^9$.

In this second example, program p chooses x^9, although student j has a higher score and claims more privileges than those claimed in x^9.

Notice that the example above shows that allocations produced using the Brazil Reserves choice function may not be fair.

The last example we give relates to the incentive-compatibility property of mechanisms.

Example 3 (Incentive-Compatibility)**.** There is one program p with a capacity of eight seats and nine students $S = \{s_1, \ldots, s_9\}$. Let $r_p = \frac{1}{2}$ and p be preferred to remaining unassigned by every student. Also, the vectors of privileges available to students are given by

$$\begin{aligned} t_{s_1} = t_{s_2} = t_{s_3} = t_{s_4} &= (p, m, i) \\ t_{s_5} = t_{s_6} &= (P, M, I) \\ t_{s_7} &= (P, m, i) \\ t_{s_8} &= (P, M, i) \\ t_{s_9} &= (P, m, I) \end{aligned}$$

In this problem, if every student claims all of the privileges that they are eligible for, there is only one stable allocation, X', that we can achieve when program p uses one of the implementations of the Brazil Reserves. The set of students assigned to p will then be the following:

$$s(X') = \{s_1, s_2, s_3, s_4, s_5, s_7, s_8, s_9\}$$

Now, assuming that the other students use the same strategy as before, if s_6 claims only public HS privilege and submits $(s_6, p, (P, m, i))$, there is again only one stable allocation, X'', that we can achieve if the program p uses one of the implementations of the Brazil Reserves. The set of students assigned to p is then the following:

$$s(X'') = \{s_1, s_2, s_3, s_4, s_5, s_6, s_8, s_9\}$$

Therefore, any stable mechanism which uses the examples of the Brazil Reserves described in section 4.1 is not incentive-compatible.

The example above shows that since these choice functions give priority to students who claim a subset of the privileges that s_6 is eligible to claim for some of the seats available, they may give student s_6 an incentive not to claim all of her privileges. This puts a burden on students to gather information about their peers and strategize their behavior in order to obtain better assignments, and gives some students an unfair advantage in their college applications, as well. It also makes it harder to observe the effect of this affirmative action policy for future decisions over it, since the privileges reported may not reflect their true demographics.

5 Empirical Evidence

Although in the last section we showed that under the government's guidelines students may not be accepted to a program unless they strategize over the privileges that they claim, one may wonder how empirically relevant those situations are. After all, when there are more candidates than seats (which is the case in the vast majority of federal programs in Brazil,) in order for a student to successfully manipulate her claims she must have an exam score that is higher than that of a student who is accepted but did not claim the privilege that she decides not to claim. For example, a low-income minority student who is not being accepted when claiming all of her eligible privileges (and therefore has an exam grade that is lower than all those who claim those privileges) can only successfully manipulate her claims if her exam score is higher than some student who is being accepted despite claiming less privileges. Since the affirmative action program is implemented to increase the access of those students when compared to a system that selects simply based on exam grades, it seems reasonable to expect these opportunities of manipulation to be rare. Note, moreover, that these allocations, which would allow for a successful manipulation of privileges claimed, are not fair: a student who is able to profitably manipulate is not accepted to a program and has a higher exam grade than a student who is accepted and is claiming less privileges.

We obtained the cut-off exam grades (that is, the lowest exam grade among those accepted to the program) for each of the five sets of seats described in section 2 for all the 3,110 federal higher education programs that participated in the SiSU in 2013 and implemented the guidelines described in section 2.[10] Following the timeline specified in the law, during this first year of implementation of the new policies, the universities could opt to allocate only a quarter of the seats that will ultimately be allocated for the affirmative action policy. That is, instead of 50% of the seats in each program, the universities could opt to offer 12.5% or more. The ratios of those seats reserved for students claiming low income and minority privileges, however, remain at 0.5 and r_p respectively.

Let $z_p^*\left(t^P, t^m, t^i\right)$ be the cutoff grade at program p for the set of seats designated for students who claim the vector of privileges $\left(t^P, t^m, t^i\right)$. For example, $z_p^*\left(P, m, I\right)$ is the cutoff grade at program p for the seats designated for students who claim public HS and low-income privileges. A necessary condition for a student to be able to successfully manipulate her claimed privileges is that the cutoff grade for seats designated with a certain set of privileges is higher than the cutoff grade for seats designated with a subset of those privileges. This means that there may be a student who was not accepted but has an exam grade high enough to be accepted when applying to a set of seats designated for a lower number of privileges. For example, suppose that there is a program $p \in P$ such that $z_p^*\left(P, m, I\right) > z_p^*\left(P, m, i\right)$. Let there be a student s with a vector of available privileges $t_s = \left(P, m, I\right)$ and exam grade $z_p\left(s\right)$ such that $z_p^*\left(P, m, I\right) > z_p\left(s\right) > z_p^*\left(P, m, i\right)$. If she claims all of her available privileges she

[10] Although the SiSU centrally matches students to programs, using a deferred acceptance procedure in which students' reported preferences are restricted to only two acceptable programs, for the sake of the analysis presented in this section those details are unimportant, since the conditions for the manipulation of the privileges claimed that we argue here, assuming a decentralized system, translate immediately to manipulations in the SiSU.

will not be accepted, since $z_p^*(P,m,I) > z_p(s)$. However, if she does not claim low-income privilege she will be accepted, since $z_p(s) > z_p^*(P,m,i)$.

Since we do not have data on the grades of the students who were not accepted, we are not able to determine whether there are, in fact, students who could have been accepted if they had manipulated the privileges they were claiming. However, given the high competition for seats in those programs — in total there were 1,757,399 candidates and 129,319 seats, an average of 13.59 candidates per seat (G1, 2013) — it is reasonable to use the occurrence of those disparities in cutoff grades as an indication of the existence of opportunities for manipulation. We therefore looked for instances in which the cutoff grades for a set of seats reserved for students claiming a certain set of privileges is higher than the cutoff grade for seats, in the same program, reserved for students who claim a subset of those privileges. The results are presented in Table 1.

	Number of occurrences (out of 3,187)	Average difference (standard deviation)
$z_p^*(P,M,I) > z_p^*(P,M,i)$	935	11.56 (13.24)
$z_p^*(P,M,I) > z_p^*(P,m,I)$	398	12.60 (14.70)
$z_p^*(P,M,I) > z_p^*(P,m,i)$	161	13.67 (15.19)
$z_p^*(P,M,I) > z_p^*(p,m,i)$	51	8.88 (8.53)
$z_p^*(P,M,i) > z_p^*(P,m,i)$	217	14.85 (17.29)
$z_p^*(P,M,i) > z_p^*(p,m,i)$	79	13.20 (12.25)
$z_p^*(P,m,I) > z_p^*(P,m,i)$	452	15.19 (16.29)
$z_p^*(P,m,I) > z_p^*(p,m,i)$	181	12.15 (13.25)
$z_p^*(P,m,i) > z_p^*(p,m,i)$	384	13.06 13.79
Number of programs with at least one of the cases above	1730 (out of 3187)	

Table 1: Instances in which the observable conditions for manipulability of the current guidelines are met and the average difference in the cutoff grades. Source: Brazilian Ministry of Education

The first fact to note is how pervasive the issue is. In more than 54% of the programs there is at least one instance in which those conditions for manipulability are observed. That is, there is a reasonable chance that in those programs, students that are the target of the affirmative action policies are not being accepted even though they have higher exam grades than those who are are accepted. One might wonder how significant are the differences between the cutoff grades presented above, since when these are too small it becomes less likely that some student would have the opportunity to successfully manipulate her outcome. The grades obtained in the exam, across all students who took it, range from 261.33 to 971.5. Since the competition for seats is very high, however, the more relevant information is how those differences compare to the range of grades that allow for acceptance in some programs. By observing the distribution of cutoff grades across all programs we therefore have a better idea of the range of exam grades obtained by those who are closer to the borderline between being accepted or not. Table 2 shows the difference between the 5% quantile and the 95% quantile, for each 90% of them are in the 500–750 range, and 61.14% are in the 600–700 range. Moreover, 64.67% of the differences summarized in Table 1 are greater than or equal to 5 points.

Seats	$z_p^*(P, M, I)$	$z_p^*(P, M, i)$	$z_p^*(P, m, I)$	$z_p^*(P, m, i)$	$z_p^*(p, m, i)$
Average	628.18	634.59	639.28	652.02	665.76
5% Quantile	563.02	564.67	572.46	578.45	593.82
95% Quantile	703.14	718.92	722.29	738.06	752.36
Difference	140.12	154.25	149.84	159.61	158.54

Table 2: Quantiles of cutoff grades

Although we could not find any data on the number of candidates and seats for each of the programs above, we did discover some information for one of the universities, UNIFESP, which published the number of candidates per seat for its 56 programs, 38 of them among those with the issue above. Table 3 shows the values of the cutoff grades and the number of candidates per seat for each type of seat for four programs in that university. These give an indication of the likely reason why the numbers in Table 1 are so dramatic: the competition for seats reserved under the affirmative action policy is very high, and therefore there are enough students who claim those privileges and have high exam grades to push up the value level of the cutoff grades. Since the number of seats allocated for affirmative action will increase to its target 50% in the following years, the proportion of programs with this issue will likely be reduced. But given how extreme the differences in how competitive the seats are, it is reasonable to expect it to remain significant.

Seats	Q_{mi}		Q_{Mi}		Q_{mI}		Q_{MI}		Q_{-}	
	Cutoff	C/S	Cutoff	C/S	Cutoff	C/S	Cutoff	C/S	Cutoff	C/S
Philosophy	652.76	28.50	671.53	18.00	657.70	25.50	688.18	30.50	675.93	10.69
History	684.29	71.00	667.92	36.67	669.67	42.50	678.29	50.00	685.78	19.00
Economics	682.82	83.50	732.68	111.00	696.24	60.00	719.46	117.00	719.26	41.65
Pharmacy	681.58	88.67	679.82	81.40	673.94	70.67	703.66	105.00	704.88	30.01

Table 3: Cutoff grades and candidates per seat (C/S) for programs at UNIFESP in 2013. Source: UNIFESP

6 Student Optimal Stable Mechanism

6.1 The multidimensional Brazil privileges choice function

One of our objectives is to find a choice function which satisfies the affirmative action objectives for each program, removes incentives for students to strategize over the privileges that they claim and guarantees the existence of a stable allocation. It could be used by the universities even in the absence of a mechanism to produce assignments. We also aim to design a mechanism that carries out our choice function's properties and finds a stable allocation.

We are proposing a new choice function, which consists of a choice procedure with slot-specific priorities, where the priorities are designed in such a way that any possible gain from strategizing over the privileges claimed is removed. It also satisfies the affirmative action objectives.

The intuition behind the way in which the slot-specific priorities are designed is that whenever a set of contracts X_t', ordered by exam grade, are in a slot's priority ordering, contracts $X_{t'}$ claiming more privileges (that is, $t' > t$) must either have a higher priority than those in X_t or must be ordered by grade together with X_t.

For example, suppose that a program p has a single seat, and can accept contracts claiming the vectors of privileges (P, M, I), (P, m, I) and (P, m, i). Priorities between these contracts are as follows:

$$X_{(p,m,i)} \succ_p X_{(P,m,I)} \succ_p X_{(P,M,I)}$$

Priorities among contracts with the same privilege vector are determined by the students' exam grades, as in section 4.1. Under those priorities, a student claiming the vector of privileges (P, M, I) would only be accepted if there were no students claiming (p, m, i) or (P, m, I), regardless of their exam grades. If that student instead claims (p, m, i) and her exam grade is high enough then she could be accepted to that seat. Consider instead the following two alternative priorities:

$$X_{(P,M,I)} \succ_p' X_{(P,m,I)} \succ_p' X_{(p,m,i)}$$

$$X_{(P,M,I)} \cup X_{(P,m,I)} \succ_p'' X_{(p,m,i)}$$

In both cases a student who is not chosen while claiming the vector of privileges (P,M,I) would not be chosen by claiming less privileges. Notice, however, that under $\succ'_p$ whenever there is at least one student claiming the vector (P,M,I) the chosen student will be one claiming that vector, whereas under $\succ''_p$ a student claiming the vector (P,M,I) will only be chosen if her exam grade is greater than all students claiming (P,M,I) or (P,m,I) in X. Therefore, if it is always the case that black students have lower exam grades than non-blacks, black students would never be chosen under $\succ''_p$.

Let $q^{MI} = q^{Mi} = \left\lceil \frac{q_p}{4} r_p \right\rceil$, $q^{mI} = q^{mi} = \left\lceil \frac{q_p}{4}(1-r_p) \right\rceil$ and $q^- = Q - 2\left(q^{MI} + q^{mi}\right)$. Given the set of valid contracts X, the multidimensional Brazil privileges choice function, denoted C^{MCF}, consists of a slot-specific priorities choice function as follows:

- A set Q^{MI}, where $\left|Q^{MI}\right| = q^{MI}$ of slots with priorities $X_{(P,M,I)} \succ^{MI} X_{(P,m,I)} \succ^{MI} X_{(P,M,i)} \succ^{MI} X_{(P,m,i)} \succ^{MI} X_{(p,m,i)}$,
- A set Q^{Mi}, where $\left|Q^{Mi}\right| = q^{Mi}$ of slots with priorities $X_{(P,M,I)} \cup X_{(P,M,i)} \succ^{Mi} X_{(P,m,I)} \succ^{Mi} X_{(P,m,i)} \succ^{Mi} X_{(p,m,i)}$,
- A set Q^{mI}, where $\left|Q^{mI}\right| = q^{mI}$ of slots with priorities $X_{(P,M,I)} \cup X_{(P,m,I)} \succ^{mI} X_{(P,M,i)} \succ^{mI} X_{(P,m,i)} \succ^{mI} X_{(p,m,i)}$,
- A set Q^{mi}, where $|Q^{mi}| = q^{mi}$ of slots with priorities $X_{(P,M,I)} \cup X_{(P,m,I)} \cup X_{(P,M,i)} \cup X_{(P,m,i)} \succ^{mi} X_{(p,m,i)}$,
- A set Q^-, where $|Q^-| = q^-$ of slots where priorities are based only on exam grades.

Contracts involving the same student s and program p but different privilege vectors, when not defined as strict by the definitions above, satisfy the ordering below:

$$(s,p,(P,M,I)) \succ (s,p,(P,m,I)) \succ (s,p,(P,M,i)) \succ (s,p,(P,m,i)) \succ (s,p,(p,m,I))$$

The precedence order in which those slots are filled is left as a choice for the policymaker. Although the order that is chosen does not impact any of the results presented in this paper, different orders of precedence may lead to accepting different sets of students.

6.2 Privilege monotonicity, fairness and affirmative action objectives

Unlike the Brazil Reserves, the choice function above gives students no incentive to leave unclaimed a privilege associated with a group she belongs to. This property will have an important role in the strategic properties of the mechanism we suggest.

Proposition 1. *The choice function C^{MCF} is privilege monotonic.*

Moreover, it also satisfies our fairness criterion:

Proposition 2. *The choice function C^{MCF} is fair.*

Regarding the satisfaction of the affirmative action objectives, however, the C^{MCF} choice function may fail to satisfy them, as defined in section 3.1 and as shown in the example below.

Example 4 (Affirmative Action Objectives)**.** For a given program p let $q_p = 8$, $r_p = \frac{1}{2}$ and let the set of contracts be $Y = \{x^1, \ldots, x^9\}$ such that $x^1_T = x^2_T = x^3_T = x^4_T = (p, m, i)$, $x^5_T = (P, m, i)$, $x^6_T = x^7_T = (P, m, I)$ and $x^8_T = x^9_T = (P, M, i)$. Also let $z_p(x^i_S) > z_p(x^j_S) \iff i < j$. The affirmative action objectives are feasible, since the set $\{x^1, x^2, x^3, x^4, x^6, x^7, x^8, x^9\}$ satisfies them. However, it is easy to see that the contracts $\{x^1_T, x^2_T, x^3_T, x^4_T x^5_T\}$ will be chosen by the choice function C^{MCF}, for any order of precedence. Since no set of three contracts in $\{x^6, x^7, x^8, x^9\}$ satisfies the requirement of having at least two low income and minorities, no instance of the C^{MCF} satisfies the affirmative action objectives.

The following weakening of the satisfaction of affirmative action includes a condition on the number of contracts claiming all privileges. This conditional satisfaction of the affirmative action requires that their objectives are fulfilled only in situations where there are enough applications claiming all three privileges, as well as, of course, that the satisfaction of all affirmative action objectives is possible.

Definition 7. A choice function $C_p : 2^X \to 2^X$ **satisfies the affirmative action objectives conditional on** q^{MI} at program p if $\forall Y \subset X$:

$$\begin{aligned} |\{x &\in Y : x_T = (P, M, I)\}| \geqslant q^{MI} \text{ implies} \\ |\{x &\in C_p(Y) : x_T \geqslant (P, m, i)\}| \geqslant \min\{\frac{q_p}{2}, |\{x \in Y : x_T \geqslant (P, m, i)\}|\}, \\ |\{x &\in C_p(Y) : x_T \geqslant (P, m, I)\}| \geqslant \min\{\frac{q_p}{4}, |\{x \in Y : x_T \geqslant (P, m, I)\}|\}, \\ \text{and } |\{x &\in C_p(Y) : x_T \geqslant (P, M, i)\}| \geqslant \min\{\frac{r_p q_p}{2}, |\{x \in Y : x_T \geqslant (P, M, i)\}|\}. \end{aligned}$$

In fact, the example below shows that even this weaker condition is not satisfied by the Brazil Reserves.

Example 5 (Affirmative Action Objectives conditional on q^{MI})**.** For a given program p let $q_p = 8$, $r_p = \frac{1}{2}$ and let the set of contracts be $Y = \{x^1, \ldots, x^9\}$ such that $x^1_T = x^2_T = x^3_T = x^4_T = (p, m, i)$, $x^5_T = x^6_T = (P, m, i)$, $x^7_T = x^8_T = (P, M, I)$ and $x^9_T = (P, m, I)$. Also, let $z_p(x^i_S) > z_p(x^j_S) \iff i < j$. If the set of contracts is Y, no matter which example of the Brazil Reserves program p uses, the chosen set will be:

$$C_p(Y) = \{x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^9\}$$

Therefore, the choice function chooses only one student claiming minority and public HS privileges, failing to satisfy the desired proportion of minority students.

The failure that the Brazil Reserves presents in satisfying the affirmative action objectives comes from the fact that it considers students claiming only public HS privilege as first in line after those claiming only minority and public HS privileges. Therefore, when there are no applications from contracts with privilege vector (P, M, i), the choice function turns to contracts with privilege vector (P, m, i) and ignores the priority for minorities. In the example above, one of the students claiming only public HS privilege receives the seat prioritized for those claiming minority and public HS privileges. Hence, implementations of the Brazil Reserves fail to satisfy the affirmative action objectives conditional on q^{MI}. This is not the case for the choice procedure we propose:

Proposition 3. *The choice function C^{MCF} satisfies the affirmative action objectives conditional on q^{MI} in any program p.*

6.3 The student optimal stable mechanism

Below we provide a brief description of the cumulative offer process, which is used to produce the student optimal stable matching. The cumulative offer algorithm and its properties are described in Hatfield and Kojima (2010). First, students submit a vector of privileges they want to claim and preferences $\succ^*$. We then use the student-proposing cumulative offer algorithm with the submitted vector of privileges $(t^s)_{s \in S}$, preferences over contracts $\succ$, which are derived from $\succ^*$ as described in section 3, and C^{MCF} for each program.

Step 1: One randomly selected student s_1 offers her most preferred contract x^1 with the vector of privileges (t^{s_1}), according to her preferences $\succ_{s_1}$. The program that receives the offer, $p_1 = x^1_P$, holds the contract. Let $A_{p_1}(1) = x^1$, and $A_p(1) = \varnothing$ for all $p \neq p_1$.

In general,

Step $k \geqslant 2$: One of the students for whom no contract is currently held by a program, say s_k, offers the most preferred contract with the vector of privileges (t^{s_k}), according to her preferences $\succ_{s_k}$, that has not been rejected in previous steps. Let us call the new offered contract, x^k. Let $p_k = x^k_P$ hold $C_{p_k}(A_{p_k}(k-1) \cup \{x^k\})$ and reject all other contracts in $A_{p_k}(k-1) \cup \{x^k\}$. Let $A_{p_k}(k) = A_{p_k}(k-1) \cup \{x^k\}$, and $A_p(k) = A_p(k-1)$ for all $p \neq p_k$.

The algorithm terminates when either every student is matched to a program or every unmatched student has no contract left to offer. The algorithm terminates in a finite number K of steps due to there being a finite number of contracts. At that point, the algorithm produces an allocation $X' = \bigcup_{p \in P} C_p(A_p(K))$, i.e., the set of contracts that are held by some program at the terminal step K.

Although we have shown that the choice function that we proposed satisfies the desired fairness and incentives properties, we are also interested in knowing whether the corresponding properties are satisfied by the overall allocation when the SOSM mechanism is used to match students to programs. The first such property that we analyze is that of fairness.

Proposition 4. *The student optimal stable mechanism, ψ^{SOSM}, is fair.*

The next property that we present here is the incentive-compatibility of the mechanism, which is a desired characteristic in mechanism design. Incentive-compatibility in this context can be described as a property that guarantees that students cannot be better off by strategizing over the preferences being submitted or privileges being claimed. In our problem, the students' strategy spaces consist not only of preferences over schools but also the privileges claimed. Although it is tempting to conclude that the incentive-compatibility of the SOSM immediately follows as a corollary of the well-known incentive properties of the SOSM mechanism, due to the wider strategy space for students the result must be obtained explicitly.

Proposition 5. *The Student Optimal Stable Mechanism, ψ^{SOSM}, is incentive-compatible.*

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Appendix

Proof of Proposition 1

Proof. Suppose, for the sake of contradiction, that there is a set of contracts $Y \subset X$, and a student s with no contract in Y, where $(s,p,t_s) \notin C_p(Y \cup \{(s,p,t_s)\})$ and $(s,p,t') \in C_p(Y \cup \{(s,p,t')\})$, for some $t' \leqslant t_s$. Since the only difference between the two sets are the contracts (s,p,t_s) and (s,p,t'), it must be the case that the contract (s,p,t') has a higher priority than (s,p,t_s) at some slot. Given the definition of the priority orderings $\succ^{MI}$, $\succ^{Mi}$, $\succ^{mI}$, $\succ^{mi}$ and $\succ^{-}$, we have that $(s,p,t_s) \succ^{MI} (s,p,t')$, $(s,p,t_s) \succeq^{mI} (s,p,t')$, $(s,p,t_s) \succeq^{Mi} (s,p,t')$, $(s,p,t_s) \succeq^{mi} (s,p,t')$ and $(s,p,t_s) \sim^{-} (s,p,t')$. Contradiction.

Hence, C_p^{MCF} is privilege monotonic. □

Proof of Proposition 2

Proof. For any set of contracts Y, the owner of any rejected contract x such that $x_T = (P,M,I)$ has lower score than the owners of the chosen contracts. So, $x \notin C_p^{MCF}(Y)$ and $x_T = (P,M,I) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$.

For any rejected contract x such that $x_T = (P,m,I)$, the only possible two types of contracts that are chosen with a lower score than x are contracts with the privilege vector (P,M,I) or (P,M,i). But, since $x_T \ngeqslant (P,M,I)$, $x_T \ngeqslant (P,M,i)$ and owners of other chosen contracts have higher scores than the owner of x, we have $x \notin C_p^{MCF}(Y)$ and $x_T = (P,m,I) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeqslant y_T \geqslant (P,m,i)$.

For any rejected contract x such that $x_T = (P,M,i)$, the only possible two types of contracts that are chosen with a lower score than x are contracts with the privilege vector (P,M,I) or (P,m,I). But, since $x_T \ngeqslant (P,M,I)$, $x_T \ngeqslant (P,m,I)$ and owners of other chosen contracts have a higher score than owner of x, we have $x \notin C_p^{MCF}(Y)$ and $x_T = (P,M,i) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeqslant y_T \geqslant (P,m,i)$.

For any rejected contract x such that $x_T = (P,m,i)$, the only possible types of contracts that are chosen with a lower score than x are contracts with the privilege vector $(P,M,I), (P,M,i)$ or (P,m,I). But, since $x_T \ngeqslant (P,M,I)$, $x_T \ngeqslant (P,M,i)$, $x_T \ngeqslant (P,m,I)$ and owners of other chosen contracts have a higher score than owner of x, we have $x \notin C_p^{MCF}(Y)$ and $x_T = (P,m,i) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeqslant y_T \geqslant (P,m,i)$.

For any rejected contract such that $x_T \ngeqslant (P,m,i)$, owners of chosen contracts with a privilege vector greater than or equal to (P,m,i) may have a lower score than the owner of x. Also, owners of other chosen contracts have a higher score than the owner of x. Therefore, we have $x \notin C_p^{MCF}(Y)$ and $x_T = \ngeqslant (P,m,i) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeqslant y_T \geqslant (P,m,i)$. Hence, for any type of contract, $x \notin C_p^{MCF}(Y) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeqslant y_T \geqslant (P,m,i)$. □

Proof of Proposition 3

Proof. For a given program p and given a set of contracts Y, let

$$|\{x \in Y : x_T = (P, M, I)\}| \geqslant q^{MI}.$$

Due to the fact that the seats in Q^{MI} give priority to contracts with the privilege vector (P, M, I) over any other type of contracts, under no circumstance less than $q^{MI} = \left\lceil \frac{q_p}{4} r_p \right\rceil$ will contracts with that type be accepted. Since all $\left\lceil \frac{q_p}{4} r_p \right\rceil$ seats in Q^{Mi} give higher priority to contracts with vector of privileges (P, M, i) or (P, M, I) over the others, the only circumstance in which less than less than $2\left\lceil \frac{q_p}{4} r_p \right\rceil$ contracts with those vectors of privileges are accepted is if $\left\lceil \frac{q_p}{4} r_p \right\rceil \leqslant |\{x \in Y : x_T \geqslant (P, M, i)\}| < 2\left\lceil \frac{q_p}{4} r_p \right\rceil$, and even in that case all those contracts will be accepted. Since $2\left\lceil \frac{q_p}{4} r_p \right\rceil \geqslant \frac{r_p q_p}{2}$, it follows that:

$$|\{x \in C_p(Y) : x_T \geqslant (P, M, i)\}| \geqslant \min\{\frac{r_p q_p}{2}, |\{x \in Y : x_T \geqslant (P, M, i)\}|\}$$

Moreover, since all $\left\lceil \frac{q_p}{4} (1 - r_p) \right\rceil$ seats in Q^{mI} give higher priority to contracts with the vector of privileges (P, m, I) or (P, M, I) over the others, the only circumstance in which less than less than $\left\lceil \frac{q_p}{4} r_p \right\rceil + \left\lceil \frac{q_p}{4} (1 - r_p) \right\rceil$ contracts with those vectors of privileges are accepted is if $\left\lceil \frac{q_p}{4} r_p \right\rceil \leqslant |\{x \in Y : x_T \geqslant (P, m, I)\}| < 2\left\lceil \frac{q_p}{4} r_p \right\rceil$, and even in that case all those contracts will be accepted. Since $\left\lceil \frac{q_p}{4} r_p \right\rceil + \left\lceil \frac{q_p}{4} (1 - r_p) \right\rceil \geqslant \frac{q_p}{4}$, it follows that:

$$|\{x \in C_p(Y) : x_T \geqslant (P, m, I)\}| \geqslant \min\{\frac{q_p}{4}, |\{x \in Y : x_T \geqslant (P, m, I)\}|\}$$

Finally, notice that all seats in Q^{MI}, Q^{Mi}, Q^{mI} and Q^{mi} , a total of $\frac{q_p}{2}$, give higher priority to contracts claiming public high school privilege. Therefore, if $|\{x \in Y : x_T \geqslant (P, m, i)\}| < \frac{q_p}{2}$, all contracts claiming that privilege will be accepted, and at least $\left\lceil \frac{q_p}{2} \right\rceil$ will be accepted otherwise. Therefore:

$$|\{x \in C_p(Y) : x_T \geqslant (P, m, i)\}| \geqslant \min\{\frac{q_p}{2}, |\{x \in Y : x_T \geqslant (P, m, i)\}|\}$$

□

Proof of Proposition 4

Proof. Assume that is not true. So, we can find $x, y \in X'$ such that $y_P >^*_{x_S} x_P$, $z_{y_P}(y_S) < z_{y_P}(x_S)$ and $x_T > y_T$. Since we have $y_P >^*_{x_S} x_P$, there exist a contract x' such that $x' = (x_S, y_P, t_{x_S})$ and $x' >^*_{x_S} x$. By the design of the cumulative offer algorithm, x' must be offered by x_S and be rejected before the final step K. Therefore, at step K, we have $y, x' \in A_{y_P}(K)$ and $X'_{y_P} = C^{MCF}_{y_P}(A_{y_P}(K))$. Since contracts are substitutes for each program and x' is rejected before the final step K, $x' \notin C^{MCF}_{y_P}(A_{y_P}(K))$ must be true. By proposition 2:

$$x' \notin C^{MCF}_{y_P}(A_{y_P}(K)) \implies z_{y_P}(y_S) > z_{y_P}(x'_S) \text{ or } x_T \nleqslant y_T$$

a contradiction. Hence ψ^{SOSM}, is fair. □

Proof of Proposition 6

Proof. For an arbitrary student s, assume that $\delta' = (t', \succ'_s) \neq (t_s, \succ_s)$. Let her assigned program from $\psi^{SOSM}(\delta', \delta_{-s})$ be p^*. Also, let δ'' be a strategy with a privilege vector t' and preference where only contract (s, p^*, t') is acceptable. Since the choice functions satisfy the substitutes condition, student s gets the same assignment from $\psi^{SOSM}(\delta'', \delta_{-s})$. This part is a corollary of Theorem 10 in Hatfield and Milgrom (2005).

Now, let δ''' be a strategy with privilege vector t_s and preference where only (s, p^*, t_s) is acceptable. Due to the privilege monotonicity of the choice functions, her assignment from $\psi^{SOSM}(\delta''', \delta_{-s})$ must be (s, p^*, t_s).

Finally, since for any given vector of privileges the choice function satisfies the substitutes condition and the Law of Aggregate Demand, we know that students cannot manipulate the student optimal stable mechanism by submitting different preferences, i.e., $\psi^{SOSM}((t_s, \succ_s), \delta_{-s}) \succeq_s \psi^{SOSM}(\delta''', \delta_{-s})$, by Theorem 11 in Hatfield and Milgrom (2005). So we have;

$$\psi^{SOSM}((t_s, \succ_s), \delta_{-s}) \succeq_s \psi^{SOSM}(\delta''', \delta_{-s}) \succeq_s \psi^{SOSM}(\delta'', \delta_{-s}) \succeq_s \psi^{SOSM}(\delta', \delta_{-s})$$

Therefore for any δ',

$$\psi^{SOSM}(\delta', \delta_{-s}) \not\succ_s \psi^{SOSM}((t_s, \succ_s), \delta_{-s})$$

Hence ψ^{SOSM} is incentive-compatible. □