Fair Implementation of Diversity in School Choice☆

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Abstract

Many school districts have objectives regarding how students of different races, ethnicity or religious backgrounds should be distributed across schools. A growing literature in mechanism design is introducing school choice mechanisms that attempt to satisfy those requirements. We show that mechanisms based on the student-proposing deferred acceptance may fail to satisfy those objectives, but that by using instead the school-proposing deferred acceptance together with a choice function used by the schools, which incorporates a preference for satisfying them, can optimally approximate the diversity objectives while still satisfying an appropriate fairness criterion. We provide analytical results which show that the proposed mechanism has a general ability to satisfy those objectives, as opposed to some currently proposed mechanisms, which may yield segregated assignments.

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JEL *classification*: C78, D63, D78, D82

1. Introduction

Over the last five decades a multitude of policies have been implemented, with varying degrees of success, to reduce historical and emerging racial, religious, and ethnic segregation at the school level. Most of the policies used to achieve that objective aim to either establish maximum quotas for the so-called majority students or to give higher priority to minority students in either all or part of the seats available.

Since the seminal work on the subject by Abdulkadiroğlu and Sönmez [5], a growing number of papers have used mechanism design principles to obtain school assignments that achieve some balance between diversity objectives, fairness, efficiency, and other properties. One class of such mechanisms, which we denote *affirmative action mechanisms*, expands the

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URL: `http://www.inaciobo.com` (Inácio Bó)

set of schools that certain types of students have access to by giving them higher priority and/or reserving some seats in the schools to be filled by those students, making the seats otherwise available to everyone.[1] One example of an affirmative action mechanism is giving higher priority to racial minorities for a number of seats in schools [17, 13, 7]. Another class of mechanisms takes diversity as an objective instead, and accommodates other properties, such as fairness or constrained efficiency. We denote that class of mechanisms *diversity implementation mechanisms*. Mechanisms with majority quotas (which enforce a maximum number of "majority type" students in each school) or others that enforce certain ratios among types of students are examples of diversity implementation mechanisms.

For problems such as university admission – which is in many cases determined by student performance in tests and high-school grades – affirmative action mechanisms could increase the diversity of cohorts by improving the access of minority students to more competitive universities. In the case of school choice, however, that it is not necessarily the case. Typically, the criteria for admission rely on aspects such as residence location, presence of siblings in the school, special needs, etc. That is, minority students are not necessarily disadvantaged with respect to others in their access to desired schools, and thus the use of such mechanisms may not help in obtaining more diverse groups of students.

We introduce a new diversity implementation mechanism that differs from others available in the literature in two main aspects: the incorporation of diversity objectives as an element of fairness and a more pragmatic interpretation of those objectives, where a given distribution of types in a school is used as a desired target instead of a strict objective.

From a theoretical perspective, one key aspect of this paper is our use of the school-proposing deferred acceptance procedure while using a choice function for the schools that incorporates a preference for groups of students that satisfy the diversity requirements. While college admissions problems are two-sided matching problems in which the welfare and incentives of both sides are under consideration, in a school choice problem the seats in the schools are simply objects to be allocated to students. Therefore the school's choice function can be designed in such a way that the property of stability and the school-optimality of the stable allocations selected *induces the desired properties on the allocation*. Moreover, as shown in section 3, the change from using the student-proposing to school-proposing deferred acceptance has significant effects on the satisfaction of diversity objectives.

1.1. Relation with the literature

The Student-Proposing Deferred Acceptance and the College-Proposing Deferred Acceptance mechanisms (SPDA and CPDA) were first introduced by Gale and Shapley [15]. While Dubins and Freedman [11] show that when using the SPDA as a direct mechanism no student or group of students can be made better-off by misrepresenting their preferences, Gale and Sotomayor [16] show that this is not normally the case when using CPDA. Furthermore, Roth

[1]The term affirmative action is normally used in a broader sense across the literature. Mechanisms such as those in Abdulkadiroğlu and Sönmez [5] and Abdulkadiroğlu [1] are denoted in those papers as implementing *affirmative action* while in our terminology those are *diversity implementation* mechanisms.

[21] shows that there is no stable mechanism that is immune to manipulation by colleges.[2]

The incentive and welfare properties of both mechanisms come into play in the context of college admissions in Balinski and Sönmez [10]. In their model there is no need for strategic or welfare considerations on the part of colleges. As a result, the SPDA is suggested as the ideal mechanism for the student placement problem.

The subsequent literature on college admissions and school choice, as well as their applications, focuses on the use of the SPDA procedure (see 5, 3, 2, and 4). When concerns about the diversity of the distribution of students across schools were introduced in the mechanisms, that choice persisted [13, 12, 14, 19, 17]. As shown in section 3, however, those mechanisms may to a great extent fail to affect the distribution of students by type in the schools. By combining the use of the CPDA procedure with a choice function used by the schools which represents a preference for satisfying the diversity objectives, we are able to obtain assignments that implement (or approximate) those objectives in a wider range of scenarios while still satisfying a fairness criterion.

The paper proceeds as follows. Section 2 introduces the model and the SPDiv mechanism. Section 3 presents the analytical results of the outcomes generated by the SPDiv mechanism and student-proposing affirmative action mechanisms. Proofs omitted from the main text can be found in the Appendix.

2. Model

A **school choice with diversity problem** consists of a tuple $\left\langle S, C, T, \tau, q, \underline{q}, \succ_S, \succ_C \right\rangle$:

1. A finite set of **students** $S = \{s_1, \ldots, s_n\}$
2. A finite set of **schools** $C = \{c_1, \ldots, c_m\}$
3. A finite set of **types** $T = \{t_1, \ldots, t_k\}$
4. A function $\tau : S \to T$ where $\tau(s)$ is the type of student s . We denote by $S^t(I)$ the set of students in $I \subseteq S$ of type t, that is, $S^t(I) = \{s \in I : t = \tau(s)\}$.
5. A capacity vector $q = (q_{c_1}, \ldots, q_{c_m})$ where q_c is the **capacity** of school $c \in C$.
6. For each school c, a vector $q_c^T = (q_c^{t_1}, \ldots, q_c^{t_k})$ of **diversity objectives**, where q_c^t is the minimum desired number of students with type t at school c, where $\sum_{t\in T} q_c^t \leq q_c$. Let $\underline{q} = \left(q_{c_1}^T, \ldots, q_{c_m}^T\right)$.
7. Students' **preference profile** $\succ_S = (\succ_{s_1}, \ldots, \succ_{s_n})$, where $\succ_s$ is a strict ranking over $C \cup \{s\}$, where s represents remaining unmatched to any school. If $s \succ_s c$, school c is deemed *unacceptable* to student s.
8. Schools' **priority profile** $\succ_C = (\succ_{c_1}, \ldots, \succ_{c_m})$, which is a collection of complete and strict rankings over the students in $S \cup \{\emptyset\}$. If $\emptyset \succ_c s$, student s is deemed *unacceptable* to school c.

[2] Whereas Dubins and Freedman [11] assume that the colleges' selection of the students can be represented by ranking them and choosing the most preferred ones up to a capacity constraint, similar results are shown for more general choice functions in, among others, Hatfield and Kojima [18] and Abdulkadiroğlu [1].

An **assignment** μ is a function from $C \cup S$ to subsets of $C \cup S \cup \{\emptyset\}$ such that: *(i)* $\mu(s) \in C \cup \{s\}$ and $|\mu(s)| = 1$ for every student s,[3] *(ii)* $|\mu(c)| \leq q_c$ and $\mu(c) \subseteq S$ for every school c and *(iii)* $\mu(s) = c$ if and only if $s \in \mu(c)$.

For a student s, $\mu(s)$ is the school to which s is assigned under μ, and for a school c, $\mu(c)$ is the set of students that are assigned to school c under μ. For a given school choice with diversity problem, we denote by $\mathcal{M}$ the set of all assignments. A set of students $I \subseteq S$ **enables diversity at school** c if for all $t \in T$, $|S^t(I)| \geq q_c^t$. An assignment μ **fully implements diversity** if for every school c, $\mu(c)$ enables diversity at c. If there is an assignment $\mu^* \in \mathcal{M}$ where μ^* fully implements diversity, we say that diversity objectives are **feasible**. We say that a student s **justifiably claims an empty seat at school** c **under the assignment** μ if s is acceptable to c, $|\mu(c)| < q_c$ and $c \succ_s \mu(s)$. An assignment μ is **non-wasteful** if no student justifiably claims an empty seat at some school. An assignment is **individually rational** if for every student s, $\mu(s) \succ_s s$ and for every school c and every student $s' \in \mu(c)$, $s' \succ_c \emptyset$. A traditionally desirable condition for an assignment to satisfy is that of having no student justifiably envying another. We define that formally, using the notion of fairness in Balinski and Sönmez [10]:

Definition 1. A student s **justifiably envies** student s' under the assignment μ, where $c = \mu(s')$, if and only if $c \succ_s \mu(s)$ and $s \succ_c s'$. An assignment μ satisfies **no justified envy** if no student justifiably envies another under μ. An assignment μ is **fair** if it is non-wasteful and satisfies no justified envy.

Although an assignment that is fair always exists [15, 10, 5], Echenique and Yenmez [12] show that an assignment that is fair and that fully implements diversity may not exist. In fact, fair assignments may completely ignore the diversity objectives, as shown in the example below:

Example 1. Consider the following school choice with diversity problem:

$$\begin{array}{rcc} S = \{s_1, s_2\} & & \\ T = \{t_1, t_2\} & \succ_{s_1}: c_2\ c_1 & \succ_{c_1}: s_2\ s_1 \\ S^{t_1}(S) = \{s_1\}, S^{t_2}(S) = \{s_2\} & \succ_{s_2}: c_2\ c_1 & \succ_{c_2}: s_2\ s_1 \\ C = \{c_1, c_2\} & & \end{array}$$

Capacities are $q_{c_1} = q_{c_2} = 1$, diversity objectives are $q_{c_1}^T = \left(q_{c_1}^{t_1}, q_{c_1}^{t_2}\right) = (0,0)$ and $q_{c_2}^T = \left(q_{c_2}^{t_1}, q_{c_2}^{t_2}\right) = (1,0)$. Consider the following assignments:

$$\mu = \begin{pmatrix} c_1 & c_2 \\ s_1 & s_2 \end{pmatrix} \qquad \mu' = \begin{pmatrix} c_1 & c_2 \\ s_2 & s_1 \end{pmatrix}$$

Diversity objectives are feasible, since the assignment μ' fully implements diversity. The unique fair assignment is μ, which doesn't fully implement diversity.

[3] We will abuse notation and consider $\mu(s)$ as an element of C, instead of a set with an element of C.

Notice that the assignment μ doesn't satisfy the diversity objective of having a student of type t_1 at school c_2 while at the same time student s_1, who is of that type, would prefer c_2 to her school under μ.

If we take the normative approach to the concept of fairness, as we do in this paper, the fairness criterion should respond to the fact that whenever the schools' priorities are "incompatible" with the diversity objectives, those objectives are ignored when using the above concept of fairness. For example, consider a city with two schools: one in a black and the other in a white neighborhood, and consider that schools give higher priority to students who live close to the school.[4] Then if the schools have objectives of having an equal distribution of white and black students, an assignment in which all black students go to one school and all white students go to the other school is fair, *even if there are students who would prefer to change their schools and join students of a different type*.[5]

A mechanism which produces assignments that eliminate those distortions must, therefore, balance the two criteria when deciding which students will be assigned to any given school: priorities and diversity objectives. A *choice function generated by reserves*, as defined by Echenique and Yenmez [12], chooses the set of students that satisfies as many diversity objectives as possible while pursuing a school's priority, and is used by schools in our mechanism. We describe it formally below.

Let $\mathbb{C}_{\mathbf{c}} : 2^S \to 2^S$ be the choice function at school c for the school choice with diversity problem $\left\langle S, C, T, \tau, q, \underline{q}, \succ_S, \succ_C \right\rangle$. Fix any $S' \subseteq S$ and let $I \subseteq S'$ be the set of students **acceptable to** c among S'. $\mathbb{C}_{\mathbf{c}}(S')$ is obtained by the following procedure.

1. **Step 0**: If $|I| \leq q_c$, $\mathbb{C}_{\mathbf{c}}(S') = I$ and the procedure ends. Otherwise if $|I| > q_c$, proceed to the steps below.
2. **Step 1**: If $|S^{t_1}(I)| < q_c^{t_1}$, accept all students in $S^{t_1}(I)$. Otherwise accept the top $q_c^{t_1}$ students in $S^{t_1}(I)$ with respect to $\succ_c$. Denote by $\Psi_{t_1}(I)$ the set of students accepted in this step.
3. **Step $\mathbf{1 < \ell \leq k}$**: If $|S^{t_\ell}(I)| < q_c^{t_\ell}$, accept all students in $S^{t_\ell}(I)$. Otherwise accept the top $q_c^{t_\ell}$ students in $S^{t_\ell}(I)$ with respect to $\succ_c$. Denote by $\Psi_{t_\ell}(I)$ the set of students accepted until step ℓ.
4. **Final step**: If $|\Psi_{t_k}(I)| < q_c$, accept the top $q_c - |\Psi_{t_k}(I)|$ students in $I \backslash \Psi_{t_k}(I)$ with respect to $\succ_c$.

The procedure above consists, therefore, of reserving q_c^t seats for each type t and filling them with the top students of that type with respect to $\succ_c$ (steps 1 to k). If there are not enough students to fill some of the reserved seats, those are made available to students of any type, together with the open seats (final step).

The notion of fairness which will result from using the choice function $\mathbb{C}_{\mathbf{c}}$ in our mechanism allows for the diversity objectives to be grounds for a kind of "justified envy" when those are

[4]For simplicity assume that only those in the white (black) neighborhood live close to the school in the white (black) neighborhood.

[5]This observation also applies to the concept of *same-type fairness*, as in Ehlers et al. [13] and Troyan and Fragiadakis [22], where an assignment is fair if there is no justified envy between students of the same type.

not being satisfied, as follows:

Definition 2. A student s **justifiably demands a seat in school** c if $c \succ_s \mu(s)$ and either: *(i)* $\left|S^{\tau(s)}(\mu(c))\right| < q_c^{\tau(s)}$, *(ii)* There is a student $s' \in \mu(c)$ such that $\tau(s') = \tau(s)$ and $s \succ_c s'$ or *(iii)* There is $t' \in T$ and $s' \in S^{t'}(\mu(c))$ such that $\left|S^{t'}(\mu(c))\right| > q_c^{t'}$ and $s \succ_c s'$.

An assignment is **fair-with-diversity** if it is individually rational, non-wasteful, and if no student justifiably demands a seat in any school.

An assignment μ is **blocked by a pair** if there are a student s and a school c such that $c \succ_s \mu(s)$ and $s \in \mathbb{C}_{\mathbf{c}}(\mu(c) \cup \{s\})$. If the choice function generated by reserves $\mathbb{C}_{\mathbf{c}}$ is used for assigning students to schools, an assignment being individually rational and not blocked by a pair is equivalent to it being fair-with-diversity, as shown in the remark below.[6]

Remark 1. Suppose that a pair (s,c) blocks an assignment μ. This means that $c \succ_s \mu(s)$ and $s \in \mathbb{C}_{\mathbf{c}}(\mu(c) \cup \{s\})$. The last condition means that either (i) $|\mu(c)| < q_c$ or (ii) $\left|S^{\tau(s)}(\mu(c))\right| < q_c^{\tau(s)}$ or (iii) $\exists\ s' \in S^{\tau(s)}(\mu(c)) : s \succ_c s'$ or (iv) $\exists s' \in \mu(c)$: $\left|S^{\tau(s')}(\mu(c))\right| > q_c^{\tau(s')}$ and $s \succ_c s'$. Since $c \succ_s \mu(c)$, condition (i) means that μ is wasteful, and condition $[(ii)$ or (iii) or $(iv)]$ means that student s justifiably demands a seat in school c.

The set of fair-with-diversity assignments is a superset of the set of fair assignments that fully implement diversity:

Proposition 1. *If μ is fair and fully implements diversity, then μ is also fair-with-diversity.*

The proof is simple and therefore omitted. Notice, however, that the assignment μ' in Example 1 is fair-with-diversity and fully implements diversity, but is not fair. The implication in Proposition 1 is, therefore, strict. The following result, however, shows that a fair-with-diversity assignment that fully implements diversity might not exist even when diversity objectives are feasible:

Proposition 2. *There might not be an assignment that is fair-with-diversity and fully implements diversity, even if diversity objectives are feasible.*

Proof. Consider the following school choice with diversity problem:

$$\begin{array}{rcc} S = \{s_1, s_2\} & & \\ T = \{t_1, t_2\} & \succ_{s_1}: c_1\ c_2 & \succ_{c_1}: s_1\ s_2 \\ S^{t_1}(S) = \{s_1\}, S^{t_2}(S) = \{s_2\} & \succ_{s_2}: c_2\ c_1 & \succ_{c_2}: s_2\ s_1 \\ C = \{c_1, c_2\} & & \end{array}$$

Capacities are $q_{c_1} = q_{c_2} = 1$, diversity objectives are $q_{c_1}^T = \left(q_{c_1}^{t_1}, q_{c_1}^{t_2}\right) = (0,0)$ and $q_{c_2}^T = \left(q_{c_2}^{t_1}, q_{c_2}^{t_2}\right) = (1,0)$. Consider the assignments μ and μ', where $\mu(c_1) = \{s_1\}$, $\mu(c_2) = \{s_2\}$,

[6] While most of the literature denotes an assignment that is individually rational and not blocked by a pair as *pairwise stable*, in this paper we refrain from using that definition, to emphasize the fact that the properties of an assignment which is fair-with-diversity constitute a normative choice of the mechanism designer, and not a characteristic of an assignment given the schools' choice functions.

$\mu'(c_1) = \{s_2\}$ and $\mu'(c_2) = \{s_1\}$. Diversity objectives are feasible, since the assignment μ' fully implements diversity. The unique fair-with-diversity assignment is μ, which doesn't fully implement diversity. □

Even though it is not always possible to obtain an assignment that fully implements diversity, we would like to have the alternative of choosing one that is as "close" to that objective as possible. In order to achieve that, we first propose the following partial order:

Definition 3. Let $>^{\underline{q}}$ be the partial order over the set of assignments $\mathcal{M}$ such that $\mu' >^{\underline{q}} \mu$ if :

1. For all $c \in C$ and $t \in T$ such that $|S^t(\mu(c))| \leq q_c^t$, $|S^t(\mu'(c))| \geq |S^t(\mu(c))|$ and
2. There are $c' \in C$ and $t' \in T$ such that $\left|S^{t'}(\mu(c'))\right| < q_{c'}^{t'}$ and $\left|S^{t'}(\mu'(c'))\right| > \left|S^{t'}(\mu(c'))\right|$.

Denote $\mu' \not>^{\underline{q}} \mu$ if $\mu' >^{\underline{q}} \mu$ is false.

In other words, $\mu' >^{\underline{q}} \mu$ if μ' has fewer seats "reserved" for students of certain types, which are occupied by students who are not of those types when compared to μ. Notice that if $\mu' >^{\underline{q}} \mu$ then it cannot be the case that both μ' and μ enable diversity in every school. As an example, suppose that there is only one school c with 100 seats and the diversity objective says that at least 50 of them should be occupied by minority students. If μ assigns 40 minority students to c and μ' assigns 45, then $\mu' >^{\underline{q}} \mu$. However, if μ'' assigns 51 minority students to c and μ''' assigns 55, both $\mu'' \not>^{\underline{q}} \mu'''$ and $\mu''' \not>^{\underline{q}} \mu''$. Increasing the number of minority students after the diversity objective is satisfied does not make an assignment greater with respect to $>^{\underline{q}}$. We now formally define a fair assignment with maximal diversity.

Definition 4. An assignment μ is **fair-with-maximal-diversity** if μ is fair-with-diversity and there is no assignment μ' such that μ' is fair-with-diversity and $\mu' >^{\underline{q}} \mu$. A mechanism is fair-with-maximal-diversity if for every school choice with diversity problem the assignment it generates is fair-with-maximal-diversity.

An assignment μ is thus fair-with-maximal-diversity if μ either fully implements diversity or μ does not fully implement diversity but there is no assignment that is fair-with-diversity and "further satisfies" some diversity objective in some school without jeopardizing another in the same or some other school.

We now proceed to show some important properties of the choice function generated by reserves $\mathbb{C}_{\mathbf{c}}$.

Definition 5. A choice function $\mathbb{C}$ satisfies the *substitutability condition* if for all $z, z' \in X$ and $Y \subseteq X$, $z \notin \mathbb{C}(Y \cup \{z\}) \implies z \notin \mathbb{C}(Y \cup \{z, z'\})$.

The condition below, together with substitutability, suffices for the existence of a matching that is individually rational and not blocked by a pair [9].

Definition 6. A choice function $\mathbb{C}$ satisfies irrelevance of rejected contracts (IRC) if $\forall I \subset S, \forall s \in S \backslash I$, $s \notin \mathbb{C}(I \cup \{s\}) \implies \mathbb{C}(I) = \mathbb{C}(I \cup \{s\})$

Echenique and Yenmez [12] show that choice rules generated by reserves satisfy substitutability and IRC:

Lemma 1 (Echenique and Yenmez [12])**.** *The function $\mathbb{C}_{\mathbf{c}}$ satisfies the substitutability condition and IRC.*

Remark 2. Let $I \subseteq S$ and $c \in C$ be such that every student in I is acceptable to school c and for every $t \in T$, $S^t(I) \geq q_c^t$. Then $\mathbb{C}_{\mathbf{c}}(I)$ enables diversity at c.

The remark above shows that whenever the set of students available is such that there are subsets of them which enable diversity at c, $\mathbb{C}_{\mathbf{c}}$ selects one of them. As we show in section 3, however, this does not guarantee that the outcome of a deferred acceptance procedure fully implements diversity even when it is feasible.

Lemma 2. *Let $|\mathbb{C}_{\mathbf{c}}(I)| < q_c$ and $I' \subseteq I$ be the set of acceptable students for school c in I. Then the following are true:* (i) $\mathbb{C}_{\mathbf{c}}(I) = I'$, (ii) $\mathbb{C}_{\mathbf{c}}(I \cup \{s\}) = \mathbb{C}_{\mathbf{c}}(I) \cup \{s\}$ *for any $s \in S$ if s is acceptable to c and (iii) $|\mathbb{C}_{\mathbf{c}}(I \cup J)| > |\mathbb{C}_{\mathbf{c}}(I)|$ for any $J \subset S$ such that for every $s \in J$, s is acceptable to c and $J \backslash I \neq \emptyset$.*

The results presented in the lemma above come easily from the definition of the procedure for $\mathbb{C}_{\mathbf{c}}$.

The School-Proposing Diversity (**SPDiv**) mechanism that we propose consists of applying the school-proposing deferred acceptance procedure described in Roth [20] using $\mathbb{C}_{\mathbf{c}}$ as the schools' choice function:

1. **Step 1:** Let $\mathbb{S}_c(1) = S$ for all $c \in C$
   (a) Each school c proposes to the students in $\mathbb{C}_{\mathbf{c}}(\mathbb{S}_c(1))$.
   (b) Each student s that received a proposal from one or more schools accepts her most preferred acceptable one according to $\succ_s$ and rejects the rest of the schools. Let, for all $c \in C$, $R_c(1)$ be the set of students who rejected school c at this step.
2. **Step k:** Let $\mathbb{S}_c(k) = \mathbb{S}_c(k-1) \backslash R_c(k-1)$ for all $c \in C$.
   (a) Each school c proposes to the students in $\mathbb{C}_{\mathbf{c}}(\mathbb{S}_c(k))$.
   (b) Each student s that received a proposal from one or more schools accepts her most preferred acceptable one according to $\succ_s$ and rejects the rest of the schools. Let, for all $c \in C$, $R_c(k)$ be the set of students who rejected school c at this step.

The procedure terminates at any step T in which no rejections are issued, and the resulting assignment μ is such that for every school c, $\mu(c) = \mathbb{C}_{\mathbf{c}}(\mathbb{S}_c(T))$ as defined above. Students who are not in the choice set of any school are left unmatched.

The following result is based on an extension of a theorem in Roth [20] for cases in which, as in this paper, choice functions are the primitives, instead of preference relations:

Lemma 3. *Suppose that students' preferences are strict and that schools use the choice function $\mathbb{C}_{\mathbf{c}}$. Then the assignment μ^C which is the outcome of the school-proposing deferred acceptance procedure is individually rational and not blocked by a pair. Moreover, μ^C is school-optimal in the sense that for each school c and assignment μ which is also individually rational and not blocked by a pair, $\mu^C(c) = \mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \mu(c)\right)$.*

Lemma 4. *Let μ and μ' be fair-with-diversity assignments. If for every $c \in C$, $\mu(c) = \mathbb{C}_{\mathbf{c}}(\mu(c) \cup \mu'(c))$ then $\mu' \not>^{q} \mu$.*

Proof. Suppose not. Then $\mu' >^{q} \mu$ and therefore there is $c \in C$ and $t \in T$ such that $|S^t(\mu(c))| < q_c^t$ and $|S^t(\mu'(c))| > |S^t(\mu(c))|$. That is, there is at least one student s of type t who is in $\mu'(c)$ but not in $\mu(c)$, implying that $\mu'(c) \setminus \mu(c) \neq \emptyset$. By individual rationality, both $\mu(c)$ and $\mu'(c)$ contain only acceptable students for c, $\mu(c) = \mathbb{C}_{\mathbf{c}}(\mu(c))$ and $\mu'(c) = \mathbb{C}_{\mathbf{c}}(\mu'(c))$.

If $|\mu(c)| < q_c$, then by Lemma 2 and the fact that $\mu'(c) \setminus \mu(c) \neq \emptyset$, $|\mathbb{C}_{\mathbf{c}}(\mu(c) \cup \mu'(c))| > |\mathbb{C}_{\mathbf{c}}(\mu(c))|$. But then $|\mathbb{C}_{\mathbf{c}}(\mu(c) \cup \mu'(c))| > |\mu(c)|$ which implies $\mu(c) \neq \mathbb{C}_{\mathbf{c}}(\mu(c) \cup \mu'(c))$, a contradiction. Thus, $|\mu(c)| = q_c$ and the procedure for $\mathbb{C}_{\mathbf{c}}(\mu(c) \cup \mu'(c))$ ends after the final step.

Since $|S^t(\mu'(c))| > |S^t(\mu(c))|$, there is at least one student $s' \in \mu'(c)$ such that $\tau(s') = t$ and $s' \notin \mu(c)$. Since $|S^t(\mu(c))| < q_c^t$, student s' is accepted in the step associated with t in $\mathbb{C}_{\mathbf{c}}(\mu(c) \cup \mu'(c))$. As a consequence, $s' \in \mathbb{C}_{\mathbf{c}}(\mu(c) \cup \mu'(c))$ and thus $\mu(c) \neq \mathbb{C}_{\mathbf{c}}(\mu(c) \cup \mu'(c))$, a contradiction. □

Putting it all together we get our main result:[7]

Theorem 1. *The SPDiv mechanism is fair-with-maximal-diversity.*

Proof. Let μ^C be the outcome of the SPDiv mechanism. By remark 1, lemmas 1 and 3, μ^C is fair-with-diversity. By lemmas 3 and 4, for any fair-with-diversity assignment μ', $\mu' \not>^{q} \mu^C$. As a consequence, μ^C is fair-with-maximal-diversity. □

3. Comparative Analysis

In order to evaluate the assignments in terms of the distribution of students across schools, we define two classes of assignments that represent the two extremes: one in which students are completely segregated by their types and one in which the distribution of types in the set of students in each school is identical to the distribution of the population as a whole. Although it is not necessarily the case that those are the designer's targets, the ability of the mechanism to attain such distribution is a good measure of how successful it is for general objectives. We now define those formally and give simple examples of both:

Definition 7. An assignment μ **maximizes segregation** if for every school $c \in C$, $s, s' \in \mu(c) \implies \tau(s) = \tau(s')$.

Definition 8. Diversity objectives **mirror the population distribution** if for every $c \in C$ and $t \in T$, $q_c^t = \left\lfloor \frac{q_c |S^t(S)|}{|S|} \right\rfloor$. An assignment μ **minimizes segregation** if μ fully implements diversity when the diversity objectives mirror the population distribution.

[7]In a version of their paper published in September 2014, Ehlers et al. [13] show simultaneous and independent work presenting this property of the school-proposing deferred acceptance mechanism.

Example 2. Suppose that $C = \{c_1, c_2\}$, $S = \{s_1, s_2, s_3, s_4\}$, $T = \{t_1, t_2\}$, $S^{t_1}(S) = \{s_1, s_2\}$ and $S^{t_2}(S) = \{s_3, s_4\}$. The assignment μ, where $\mu(c_1) = \{s_1, s_2\}$ and $\mu(c_2) = \{s_3, s_4\}$ *maximizes segregation*, while the assignment μ', where $\mu'(c_1) = \{s_1, s_3\}$ and $\mu(c_2) = \{s_2, s_4\}$, *minimizes segregation.*

We denote the partition of the students by type by $S = S_1 \cup \cdots \cup S_k$, where for every $i \in \{1, \ldots, k\}$ and $s \in S_i$, $\tau(s) = t_i$. We consider a simplified configuration in which the number of students of each type is the same, that is, $|S_i| = |S_j|$ for all i, j, every school has the same capacity q and the number of seats in schools equals the number of students ($|S| = q\,|C|$). In order to avoid issues related to fractional values throughout the analysis we will, for any given value of k (the number of types of students), assume that the number of students is such that $|S| = n_1 n_2 k^2$ and that the number of schools is such that $|C| = n_2 k$, for some $n_1, n_2 \in \mathbb{N}$. As a result, $q = n_1 k$ and for every i, $|S_i| = n_1 n_2 k$ and $\frac{q|S_i|}{|S|} = n_1$.

Our first result shows that the SPDiv mechanism generates assignments that minimize segregation regardless of students' preference profiles or schools' priorities.

Theorem 2. *Every assignment generated by the SPDiv mechanism minimizes segregation when the diversity objectives mirror the population distribution.*

We now consider two scenarios with a generalization for multiple types of students in the Deferred Acceptance with Minority Reserves (DAMR) mechanism, proposed in Hafalir et al. [17], which is essentially the student-proposing deferred acceptance version of the SPDiv mechanism.[8] The first scenario below is one in which students of each type have an exclusive set of schools that are preferred by those students to all other schools. This class of preferences accommodates, among other things, a situation that is commonly observed: students (and their parents) have a preference for schools that historically have a significant proportion of students of their own type.

Scenario 1. (Favorite schools for each type*) There is a partition of schools $C = C_1 \cup \cdots \cup C_k$ where $|C_1| = \cdots = |C_k|$ and for every student $s_i \in S_i$ and $j \neq i$ it follows that $c_i \in C_i$ and $c_j \in C_j$ implies $c_i \succ_{s_i} c_j$.*

Now we define a scenario in which the set of schools can be partitioned such that the preference among schools in different partitions is perfectly correlated across students, but not between schools in them. This is common when, for example, schools in wealthier neighborhoods are perceived as being better than those in less wealthy neighborhoods.

Scenario 2. (Tiered schools) *There is a partition of schools $C = C_1 \cup \cdots \cup C_a$, with $a \geq k$, where $|C_1| = \cdots = |C_a|$ and for every student $s \in S$, schools $c_i \in C_i$ and $c_{i+1} \in C_{i+1}$, student s's preferences are such that $c_i \succ_s c_{i+1}$.*

[8] We use a simple extension of the DAMR mechanism to accommodate for more than one type of student (17 consider only two types: minority and majority). This extension can also be found in Echenique and Yenmez [12] and is a special case of the Deferred Acceptance Procedure with Soft Bounds in Ehlers et al. [13], where there are no upper quotas. See also Kominers and Sönmez [19] and Westkamp [23] for more generalizations.

The following property connects DAMR outcomes with the set of fair-with-diversity assignments:

Lemma 5. *Every assignment μ generated by the DAMR mechanism is fair-with-diversity.*

Proof. By Lemma 1 and Theorem 1 in Aygün and Sönmez [8], μ is individually rational and is not blocked by a pair. Thus μ is fair-with-diversity. □

We now consider the application of the DAMR mechanism in those different scenarios:

Proposition 3. *Every assignment generated by the DAMR mechanism maximizes segregation in scenario 1.*

It is easy to see that the result in proposition 3 applies not only to the DAMR mechanism, but to *any mechanism that uses the student-proposing deferred acceptance procedure* when the school's choice function satisfies the *substitutability condition* and the *law of aggregate demand* (property introduced in Alkan [6]).

Proposition 4. *Let μ be an assignment generated by the DAMR mechanism when the diversity objectives mirror the population distribution, and let $C^* \subseteq C$ be the set of schools such that if $c^* \in C^*$, $\mu(c^*)$ enables diversity in c^*. Then in scenario 2 $\frac{|C^*|}{|C|} \geq \frac{a-1}{a}$.*

The lower-bound in the proposition may be binding. As a result, in a situation where there are only two types of students (minority and majority), for example, half of the schools may constitute a totally segregated subset of them, as shown in the example below.

Example 3. Consider the following school choice with diversity problem:

$$\begin{aligned} S &= \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\} \\ T &= \{t_1, t_2\} \\ S^{t_1}(S) &= \{s_1, s_2, s_3, s_4\} \\ S^{t_2}(S) &= \{s_5, s_6, s_7, s_8\} \\ C &= \{c_1, c_2, c_3, c_4\} \end{aligned} \qquad \begin{aligned} &\succ_{s_1}: c_1\ c_2\ c_3\ c_4 \\ &\succ_{s_2}: c_2\ c_1\ c_3\ c_4 \\ &\succ_{s_3}: c_1\ c_2\ c_3\ c_4 \\ &\succ_{s_4}: c_2\ c_1\ c_3\ c_4 \\ &\succ_{s_5}: c_1\ c_2\ c_3\ c_4 \\ &\succ_{s_6}: c_2\ c_1\ c_3\ c_4 \\ &\succ_{s_7}: c_1\ c_2\ c_4\ c_3 \\ &\succ_{s_8}: c_2\ c_1\ c_4\ c_3 \end{aligned} \qquad \begin{aligned} &\succ_{c_1}: s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8 \\ &\succ_{c_2}: s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8 \\ &\succ_{c_3}: s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8 \\ &\succ_{c_4}: s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8 \end{aligned}$$

Capacities are $q_c = 2$, diversity objectives are $q_{c_j}^{t_i} = 1$, for all $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\}$. The assignment generated by the DAMR mechanism is μ, as follows:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ s_1, s_5 & s_2, s_6 & s_3, s_4 & s_7, s_8 \end{pmatrix}$$

Note that both $\mu(c_1)$ and $\mu(c_2)$ enable diversity in those schools but the remaining population is segregated: to school c_3 only students of type t_1 are assigned and to school c_4 only students of type t_2.

When $a = |C|$, that is, the set of schools is partitioned such that each partition has only one school, the preferences in scenario 2 are equivalent to a situation in which all students have the same preferences among schools. It is easy to see that proposition 4 leads to the following corollary:

Corollary 1. *Every assignment generated by the DAMR mechanism minimizes segregation when all students have the same preferences.*

The value of a in scenario 4 indicates, in a way, the degree of correlation among students' preferences over schools: the larger the value of a, the more similar they are. Proposition 4 therefore shows that the DAMR mechanism may be an adequate choice of mechanism in situations where students' preferences follow, for example, a widely known ranking, but less so when preferences are more heterogeneous.

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AppendixA. Proofs

Lemma 3

This proof consists of reproducing the steps in the same result given in Roth [20] under the more general assumption that the primitives are choice functions that satisfy substitutability and IRC instead of choice functions derived from preferences over sets of students.

We make use of some definitions and the following results in Roth [20], which remain valid without assuming IRC:

Lemma 6. *[20] Let* $s^* \in S_1$ *and* $s^* \in \mathbb{C}_{\mathbf{c}}(S_1 \cup S_2)$*. Then* $s^* \in \mathbb{C}_{\mathbf{c}}(S_2 \cup \{s^*\})$*.*

Proposition 5. *[20]Offers remain open: for every school* c*, if* $s \in \mathbb{C}_{\mathbf{c}}(\mathbb{S}_c(k-1))$ *and is not rejected by student* s *in step* $k-1$*, then* $s \in \mathbb{C}_{\mathbf{c}}(\mathbb{S}_c(k))$*.*

Proposition 6. *[20]Rejections are final: If* s *rejects school* c *at step* k*, then for any* $p \geq k$ *student* s *would reject another proposal from* c*. In other words, if* $s(p)$ *is the set of schools that propose to student* s *at step* p*, she would never choose* c *out of* $\{\{s\} \cup s(p) \cup \{c\}\}$*.*

For the next steps of the proof, however, in order to not have to assume that the choice functions are derived from strict preferences over sets of students, we derive the results by assuming that the choice function satisfies IRC. We will use the following lemma, which comes easily from the definition of the IRC condition:

Lemma 7. *Let* $\mathbb{C}_{\mathbf{c}}$ *satisfy IRC and let* $X \subseteq S$ *and* $Y = \mathbb{C}_{\mathbf{c}}(X)$ *. Then, for any* $Z \subseteq X \backslash Y$*,* $\mathbb{C}_{\mathbf{c}}(Y \cup Z) = \mathbb{C}_{\mathbf{c}}(Y)$*.*

Proposition 7. *The outcome of the school-proposing deferred acceptance procedure above is individually rational and is not blocked by a pair.*

Proof. Let μ^C be the outcome of the procedure. First, note that for every student s, $\mu^C(s) \succeq_s s$, otherwise $\mu^C(s)$ would have been rejected by s. Now suppose that $\mu^C(c)$ is not individually rational, and therefore $\mathbb{C}_{\mathbf{c}}\left(\mu^C(c)\right) \neq \mu^C(c)$. Then there is s such that $s \in \mathbb{C}_{\mathbf{c}}(\mathbb{S}_c(T))$ but $s \notin \mathbb{C}_{\mathbf{c}}\left(\mu^C(c)\right)$. But since $\mu^C(c) \subseteq \mathbb{S}_c(T)$ this would violate the substitutability of $\mathbb{C}_{\mathbf{c}}$, and therefore $\mathbb{C}_{\mathbf{c}}\left(\mu^C(c)\right) = \mu^C(c)$. Now suppose that there is a student s and school c such that $c \succ_s \mu^C(s)$ and $s \in \mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \{s\}\right)$. By assumption and propositions 5 and 6 above, student s didn't reject any proposal from school c, and therefore $s \in \mathbb{S}_c(T)$. Denote $R^T \equiv \mathbb{S}_c(T) \setminus (\mathbb{C}_{\mathbf{c}}(\mathbb{S}_c(T)) \cup \{s\})$. Then $\mathbb{S}_c(T) = \mu^C(c) \cup R^T \cup \{s\}$, and therefore $\mu^C(c) = \mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup R^T \cup \{s\}\right) = \mathbb{C}_{\mathbf{c}}\left(\mu^C(c)\right)$. Since students in R^T are rejected, by IRC $\mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \{s\}\right) = \mathbb{C}_{\mathbf{c}}\left(\mu^C(c)\right)$. Contradiction with $s \in \mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \{s\}\right)$. ☐

Proposition 8. *The outcome of the procedure above,* μ^C*, is school-optimal in the sense that, for each school* c *and every assignment* μ *which is individually rational and is not blocked by a pair,* $\mu^C(c) = \mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \mu(c)\right)$*.*

Proof. Suppose not. Then there exists an assignment μ^* and a school c such that μ^* is individually rational, not blocked by a pair, and $\mu^C(c) \neq \mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \mu^*(c)\right)$. We first show that there is at least one student $s \in \mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \mu^*(c)\right)$ such that $s \notin \mu^C(c)$ and $s \in \mu^*(c)$. If that isn't the case, since $\mu^C(c) \neq \mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \mu^*(c)\right)$, it must be that $\mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \mu^*(c)\right) \subsetneq \mu^C(c)$. But then all students in $\mu^*(c)$ are rejected and by Lemma 7 $\mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \mu^*(c)\right) = \mathbb{C}_{\mathbf{c}}\left(\mu^C(c)\right) = \mu^C(c)$, where the second equality is proved in Proposition 7 and therefore constitutes a contradiction. ☐

We show that no student will reject an *achievable* school. A school c is achievable to student s if there exists an assignment μ such that μ is individually rational, is not blocked by a pair and $s \in \mu(c)$. The proof is by induction. By induction assumption, up to step

$k-1$ no student rejected any achievable school. Now suppose that student s rejects school c, which is achievable for her, in favor of another school c'. It must then be that $c' \succ_s c$. Moreover, since school c is achievable to s, there exists at least one matching μ', where μ' is individually rational and is not blocked by a pair, in which $\mu'(s) = c$. Since up to step $k-1$ no student rejected an achievable school, no student in $S \backslash \mathbb{S}_{c'}(k)$ can be in $\mu'(c')$ and therefore $\mu'(c') \subset \mathbb{S}_{c'}(k)$. But since student s rejected school c in favor of c' at step k, it must be that $s \in \mathbb{C}_{c'}(\mathbb{S}_{c'}(k))$. Since μ' is individually rational, $\mu'(c) = \mathbb{C}_{c'}(\mu'(c))$. We have, therefore, $s \notin \mathbb{C}_{c'}(\mu'(c))$, $\mu'(c') \subset \mathbb{S}_{c'}(k)$ and $s \in \mathbb{C}_{c'}(\mathbb{S}_{c'}(k))$, which is a contradiction with $\mathbb{C}_{c'}$ satisfying substitutability.

Given that, it must then be that all students who have c as achievable remain available until the last step of the algorithm and therefore $\mu^*(c) \subset \mathbb{S}_c(T)$. Since $\mu^C(c) = \mathbb{C}_{\mathbf{c}}(\mathbb{S}_c(T)) \subset \mathbb{S}_c(T)$, we can rewrite this as $\mu^C(c) = \mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \mu^*(c) \cup \left(\mu^*(c) \backslash \mu^C(c)\right) \cup \left(\mathbb{S}_c(T) \backslash \mu^C(c)\right)\right)$. By Lemma 7, $\mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \mu^*(c) \cup \left(\mu^*(c) \backslash \mu^C(c)\right) \cup \left(\mathbb{S}_c(T) \backslash \mu^C(c)\right)\right) = \mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \mu^*(c)\right)$. That is, $\mu^C(c) = \mathbb{C}_{\mathbf{c}}\left(\mu^C(c) \cup \mu^*(c)\right)$ which is a contradiction to our initial assumption.

Theorem 2

Suppose, for the sake of contradiction, that schools' diversity objectives mirror the population distribution but the assignment μ generated by the SPDiv mechanism does not minimize segregation. Then there is a school $c \in C$ and a type $t_i \in T$ such that $|S^{t_i}(\mu(c))| < n_1$. Then at some step ℓ in the deferred acceptance procedure of the SPDiv mechanism, the set of students of type t_i that rejected school c has more than $(n_1 - 1)\, n_2 k$ elements. Without loss of generality, let ℓ be the earliest step at which this happens to some school for any type (c may or may not be the only school for which that happens during step ℓ.) Since students consider all schools acceptable, all those students rejected c because another school proposed simultaneously. By Proposition 2 in Roth [20], if a student receives an offer during a step of the procedure, it may change its assignment over time, but will not become unmatched at any subsequent step. Therefore, those students who rejected c are assigned to other schools. But then at least one school $c' \in C$, with $c' \neq c$ proposed to more than n_1 students of type t_i during step ℓ. This can only happen if some student of type t_i is accepted during the final step of the procedure for the choice function $\mathbb{C}_{\mathbf{c'}}$. This implies that at the step associated with some type $t_j \neq t_i$ the number of students of type t_j that rejected c' at some step earlier than ℓ is greater than $(n_1 - 1)\, n_2 k$. Contradiction with ℓ being the earliest step at which this happened.

Proposition 3

It is sufficient to show that, for any given i, every student $s_i \in S_i$ is accepted by some school in C_i. Suppose that there is a student $s_i \in S_i$ that is not assigned to any school in C_i. Since all schools in C_i are preferred by s_i to any other schools, s_i was rejected by all schools in C_i. Since every student is acceptable by all schools and all students in S_i have the same type, every school c will simply accept the top q_c students according to $\succ_c$ or all students in case the number of those pointing to c is lower than q_c. Thus, the only way in which a student is rejected by a school is if that school has already accepted q_c students. Notice that since $\mathbb{C}_{\mathbf{c}}$ satisfies the law of aggregate demand (see Echenique and Yenmez [12]), by the end of each step of the procedure the number of accepted students in each school never decreases. Thus, by the end of step 1, since s_i was rejected by her first choice, at least q_c students in S_i were accepted by schools in C'_i. By the end of step 2, at least $2q_c$ students are accepted, since the school mentioned in step 1 will still accept q_c students by the end of step 2, and the second-best school for s_i also accepts q_c students. By repeating the argument, by the end of step $|C_i|$, at least $|C_i|\, q_c$ students were accepted in the first $|C_i|$ steps. But notice that during the first $|C_i|$ steps, students in S_j could have only pointed to schools in C_j, for any $j \in \{1, \dots, k\}$. Thus there is at least $|C_i|\, q_c + 1$ students in S_i, which is a contradiction with $\sum_{c \in C'_i} q_c = |S_i|$. Therefore, students in S_i will all be assigned to schools in C_i and thus $\mu(s) \in C_i \implies s \in S_i$.

Proposition 4

Let μ be the assignment generated by the DAMR mechanism. By Lemma 5, μ is fair-with-diversity. Therefore μ is non-wasteful and thus every school is assigned q_c students and every student is assigned to a school. We show that when diversity objectives mirror the population distribution, $\mu(c)$ enables diversity in every school $c \in C_1 \cup \dots \cup C_{a-1}$. We will prove by induction in the sets $C_1, \dots, C_{a-1}$ when $t > 1$ since the case $a = k = 1$ is trivial.

Step 1: we want to show that for every school $c \in C_1$, $\mu(c)$ enables diversity at c.

Let n_3 be the integer such that $n_3 = |C_1| = \dots = |C_a|$. Since $|C| = n_2 k$, $n_3 a = n_2 k$. And since $a \geq k$, $n_3 \leq n_2$. We must first show that there are at least $n_1 n_3$ students of each type $t \in T$ in $\bigcup_{c \in C_1} S^t(\mu(c))$. Suppose not. Then there is at least one school $c' \in C_1$ such that $|S^t(\mu(c'))| < n_1$. Since $|S^t(S)| = n_1 n_2 k \geq n_1 n_3 k$ and $k > 1$, then there is a student s' such that $\tau(s') = t$ and $\mu(s') \notin C_1$. But then $c' \succ_{s'} \mu(s')$ and s' justifiably demands a seat in c', which implies that μ isn't fair-with-diversity and thus we have a contradiction. Moreover, since there are at least $n_1 n_3$ students of each type t in $\bigcup_{c \in C_1} S^t(\mu(c))$, there are at least $n_1 n_3 k$ students in $\bigcup_{t \in T} \bigcup_{c \in C_1} S^t(\mu(c))$. Since $q_c = n_1 k$ and for any i, $|C_i| = n_3$ it follows that there are *exactly* $n_1 n_3$ students of each type t in $\bigcup_{c \in C_1} S^t(\mu(c))$.

Suppose now that there is a school $c' \in C_1$ such that $\mu(c)$ doesn't enable diversity at c'. Then there is a type $t \in T$ such that $|S^t(\mu(c'))| < n_1$. By the result above we know that there is a student s of type t such that $\mu(s) \notin C_1$. By assumption on preferences, $c' \succ_s \mu(s)$, implying that s justifiably demands a seat in c', a contradiction.

Step k^*: by induction assumption, for every $i \in \{1, \dots, k^* - 1\}$ and $c \in C_i$, $\mu(c)$ enables diversity in c. The proof follows the same argument as for step 1, with the difference that in the instances in which a student s justifiably demands a seat in some school $c' \in C_{k^*}$, that student is assigned in μ to some school in $C_{k'}$, where $k' > k^*$, implying that $c' \succ_s \mu(s)$.

Note, however, that when $k^* = a$ this argument cannot be made any longer, that is, if there is a school $c' \in C_k$ such that $|S^t(\mu(c'))| < n_1$, there may not exist a student s of type t such that $c' \succ_s \mu(s)$.