Generalized Cumulative Offer Processes*

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Abstract

In the context of the matching-with-contracts model, we generalize the cumulative offer process to allow for arbitrary subsets of doctors to make proposals in each round. We show that, under a condition on the hospitals' choice functions, the outcome of this generalized cumulative offer process is independent of the sets of doctors making proposals in each round. The flexibility of the resulting model allows it to be used to describe different dynamic processes and their final outcomes.

Keywords: Matching with contracts, cumulative offer mechanism, asynchrony, order independence.
JEL Codes: C78; D44; D47.

1 Introduction

In the domain of matching and discrete allocation problems, step-by-step procedures are often used to describe how outcomes can be constructed. Gale and Shapley (1962), for example, describe an algorithm—the deferred acceptance (DA)—that produces a matching of students to colleges by simulating a process involving a sequence of applications by students and tentative acceptances and rejections by colleges. Similarly, when considering a labor market, Crawford and Knoer (1981) and Kelso and Crawford (1982) describe processes in which firms make offers sequentially to workers, adjusting the salaries accordingly, until a stable equilibrium is reached.

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While not always emphasized in these papers, many of these algorithms could be interpreted as describing dynamic processes that can take place in the real world—students applying to schools and being "waitlisted", firms making offers to workers and adjusting salaries, etc. One obstacle in this interpretation is that the description of these algorithms often involves unrealistic sequences and timing of decisions. In DA, all students who are not held at some college make simultaneous offers. In Crawford and Knoer (1981) and Kelso and Crawford (1982), firms still looking for workers to hire also make offers simultaneously.

On the other hand, some papers describe decentralized and asynchronous matching processes. Roth and Vate (1990) show how, starting from any matching, the random satisfaction of blocking pairs always converges to a stable matching.[1] Blum *et al.* (1997) describe a senior labor market in which firms fill vacant positions by targeting desirable workers in other firms, in a process that results in "vacancy chains". While the processes described in both papers might follow an indetermined or stochastic order of proposals, they always involve only one agent doing so per period. Roth and Xing (1997) describes the entry-level market for Clinical Psychologists as a decentralized process involving, among other phases, an asynchronous and stochastic version of the DA algorithm.

In this paper, our focus is on the cumulative offer process (COP) introduced by Hatfield and Milgrom (2005). Hatfield and Milgrom (2005) extend the standard matching model to include contracts between doctors and hospitals, generalizing both the college admission and labor market models above, and introducing an algorithm that under certain conditions on the preferences of the participants, produces stable matchings. Similarly to DA, the COP involves a sequence of contractual offers by doctors, tentative acceptances, rejections and renegotiations by hospitals where (i) a new iteration takes place only after all rejections are made and (ii) all possible offers are made simultaneously in each iteration.

We show that decentralized and asynchronous matching processes can be embedded in the COP. The result has already been partially shown. The process in Hatfield and Kojima (2010) describe the process as involving an offer from a single doctor at a time. Hirata and Kasuya (2014) showed that, under certain conditions, all single-offer COPs (regardless of the order in which the doctors make their offers) induce the same outcome as the simultaneous-offer COP. Hatfield *et al.* (2021) provide an alternative order independence result for single-offer COPs. However, there is a clear gap for the COP to accommodate decentralized and asynchronous matching processes. Does it need to wait until all rejections are made? Does it need to consider all possible offers or just a single offer?

In this paper, we extend the COP to allow for arbitrary subsets of doctors to make offers in every period. This generalized COP includes both the single-offer and simultaneous-offer COPs as special cases. We obtain an "order independence" of this offer process, generalizing the result of Hirata and Kasuya (2014) to show that under the same conditions considered by these authors, any

[1] Refer to Definition 4 for the formal definition of stability.

arbitrary subset of doctors making offers in each period results in the same outcome (Theorem 1). In section 5 we establish the relation between the conditions used for order independence results in Hirata and Kasuya (2014) and Hatfield *et al.* (2021).

The generalized COP is flexible enough to allow for applications to be sent at any time and for hospitals to make decisions as they receive applications. As an example, consider the following simple scenario with three doctors, $\{d_1, d_2, d_3\}$, and two hospitals with one quota each, $\{h_1, h_2\}$. Each pair of doctor and hospital has only one potential contract. Their preferences are shown below.

d_1	d_2	d_3	h_1	h_2
h_1	h_2	h_1	d_2	d_1
h_2	h_1	h_2	d_3	d_3
			d_1	d_2

When there is only one potential contract per doctor-hospital pair, the simultaneous-offer COP *à la* Hatfield and Milgrom (2005) is identical to DA. The corresponding process is visualized below, where $d_i \to h_j$ denotes that doctor d_i's most preferred contract (among those that have not yet been reviewed) involves hospital h_j, and (h_j, d_i) denotes that h_j has reserved the contract involving d_i.

$$\begin{array}{c} (h_1, \emptyset) \\ (h_2, \emptyset) \end{array} \quad \begin{array}{c} d_1 \to h_1 \\ d_2 \to h_2 \\ d_3 \to h_1 \end{array} \quad \begin{array}{c} (h_1, d_3) \\ (h_2, d_2) \end{array} \quad d_1 \to h_2 \quad \begin{array}{c} (h_1, d_3) \\ (h_2, d_1) \end{array} \quad d_2 \to h_1 \quad \begin{array}{c} (h_1, d_2) \\ (h_2, d_1) \end{array} \quad d_3 \to h_2 \quad \begin{array}{c} (h_1, d_2) \\ (h_2, d_1) \end{array}$$

In the first round, h_1 reviews the contracts with d_1 and d_3 and reserves the contract with d_1, while h_2 reviews and reserves the contract with d_2. In the second round, d_1 has no reserved contracts and d_1's second most preferred contract involves h_2. Hospital h_2 reviews and reserves this contract, leaving d_2 without a reserved contract. The simultaneous-offer COP continues like this until each doctor has either (i) a reserved contract or (ii) no potential contracts left to be reviewed.

Hirata and Kasuya (2014) show that the single-offer COP produces the same outcome as above, i.e., it will select the same set of contracts.

Suppose instead that hospitals take turns in reviewing applications. E.g., h_1 reviews all of its applications in the first round, h_2 reviews all of its applications in the second round, h_1 reviews all of its applications in the third round, and so on. In every step, each doctor d lacking a reserved contract applies to the hospital involved in d's most preferred contract among those that have not yet been reviewed. Such a process could take place if, for example, one hospital has a more efficient administration that is capable of processing applications at a faster pace. After reviewing its initial batch of applications, the first hospital necessarily needs to wait for the second hospital to review its applications before it can take any further action. As illustrated below, this process will result in the same set of contracts regardless of which hospital is first to evaluate its applications.

$$\begin{matrix}(h_1,\emptyset)\\(h_2,\emptyset)\end{matrix}\;\begin{matrix}d_1\to h_1\\d_3\to h_1\end{matrix}\;\begin{matrix}(h_1,d_3)\\(h_2,\emptyset)\end{matrix}\;\begin{matrix}d_1\to h_2\\d_2\to h_2\end{matrix}\;\begin{matrix}(h_1,d_3)\\(h_2,d_1)\end{matrix}\;d_2\to h_1\;\begin{matrix}(h_1,d_2)\\(h_2,d_1)\end{matrix}\;d_3\to h_2\;\begin{matrix}(h_1,d_2)\\(h_2,d_1)\end{matrix}$$

$$\begin{matrix}(h_1,\emptyset)\\(h_2,\emptyset)\end{matrix}\;d_2\to h_2\;\begin{matrix}(h_1,\emptyset)\\(h_2,d_2)\end{matrix}\;\begin{matrix}d_1\to h_1\\d_3\to h_1\end{matrix}\;\begin{matrix}(h_1,d_3)\\(h_2,d_2)\end{matrix}\;d_1\to h_2\;\begin{matrix}(h_1,d_3)\\(h_2,d_1)\end{matrix}\;d_2\to h_1\;\begin{matrix}(h_1,d_2)\\(h_2,d_1)\end{matrix}\;d_3\to h_2\;\begin{matrix}(h_1,d_2)\\(h_2,d_1)\end{matrix}$$

Note that (i) neither of these processes can occur in a simultaneous-offer or single-offer COP and (ii) both processes produce the same outcome as the simultaneous-offer and single-offer COPs, suggesting that synchronization is unnecessary.

More than providing an additional family of algorithms for computing stable matchings with contracts (see Corollary 1), the generalized COP can be used to describe more realistic dynamic processes. For example, it can be used to describe a process in which doctors make offers to hospitals asynchronously, and these process pending proposals asynchronously as well. This includes processes in which, whenever a contract is rejected, the most preferred remaining contract of the rejected doctor could immediately be made available for consideration by another hospital. Such processes can arise in applications that are susceptible to delays in decision making, but can not be described by single-offer or simultaneous-offer COPs. Single-offer COPs require offers to be evaluated in order, one at a time, while the simultaneous-offer COP requires all hospitals to finish evaluating their current offers in each round before any hospital can move on to evaluate the offers in the next round. The generalized COP does not require coordination of this kind and Theorem 1 shows that a lack of synchrony in these decisions is inconsequential to the final outcome. Section 6 concludes with brief descriptions of other applications for the model. All proofs can be found in Appendix A.

2 Preliminaries

Let D be a finite set of doctors, let H be a finite set of hospitals, and let $X \subseteq D \times H \times \Theta$ be a finite set of contracts where Θ is a finite set (e.g., wages and job descriptions), with $d \in D$, $h \in H$, $x \in X$, and $\theta \in \Theta$ being their typical elements. For each contract $x \in X$, let $d(x)$ and $h(x)$ denote the doctor and hospital involved in x, respectively. For any $X' \subseteq X$, let $X'_i \coloneqq \{x \in X' \mid i \in \{d(x), h(x)\}\}$ for every $i \in D \cup H$. For any $X' \subseteq X$, let $d(X') \coloneqq \bigcup_{x\in X'}\{d(x)\}$ and $h(X') \coloneqq \bigcup_{x\in X'}\{h(x)\}$. We call $X' \subseteq X$ an *allocation* if $|X'_d| \leq 1$ for each $d \in D$. At an allocation X', each doctor in $D \setminus d(X')$ is assigned the *null contract* $x^{\emptyset}$.

For each doctor $d \in D$, let $\succ_d$ be the doctor's strict preference relation over $X_d \cup \{x^{\emptyset}\}$ and $\mathcal{P}_d$

be the set of all possible strict preference relations over $X_d \cup \{x^\emptyset\}$. Let $\mathcal{P}_D \coloneqq \prod_{d\in D} \mathcal{P}_d$ be the set of all possible preference profiles, with $\succ_D \in \mathcal{P}_D$ being a typical element of $\mathcal{P}_D$. A contract $x \in X_d$ is *acceptable* to doctor d if $x \succ_d x^\emptyset$. Let $AC(\succ_d) \coloneqq \{x \in X_d \mid x \succ_d x^\emptyset\}$ be the set of acceptable contracts to a doctor with preference relation $\succ_d$. We assume that $|AC(\succ_d)| \geq 1$ for each $\succ_d \in \mathcal{P}_d$ and each $d \in D$.

Each hospital $h \in H$ has a choice function $C_h : 2^X \to 2^{X_h}$, such that for any $X' \subseteq X$, $C_h(X') \subseteq X'_h$. For each $h \in H$, C_h chooses at most one contract for each $d \in D$; that is, for any $X' \subseteq X$ and any $h \in H$, $C_h(X')$ is an allocation. Let $C_H = (C_h)_{h\in H}$ be a profile of hospitals' choice functions. For any $X' \subseteq X$ evaluated by hospital h, h's choice function could take into account the set of contracts not involving h, denoted by $X'_{-h} \subseteq X'$. This possibility is, however, ruled out by the common and widely accepted assumption that choice functions satisfy irrelevance of rejected contracts (see Lemma 1 below).

Definition 1 (Aygün and Sönmez (2013))**.** *Hospital h's choice function C_h satisfies the **irrelevance of rejected contracts** (IRC) condition if for any $X' \subset X$ and $x \in X$,*[2] *if $x \notin C_h(X' \cup \{x\})$, then $C_h(X') = C_h(X' \cup \{x\})$.*

Note that Definition 1 does not require contract x to involve hospital h.

Lemma 1. *Suppose that each hospital h's choice function satisfies the IRC condition. Then for each $h \in H$ and each $X' \subseteq X$, $C_h(X') = C_h(X'_h)$.*

That is, the IRC condition implies that the choice of a hospital h is only affected by the contracts involving h. Throughout the paper, we assume that all choice functions satisfy the IRC condition.[3]

3 Generalized Offer Process

In this section, we introduce a generalized COP, or the GCOP. It generalizes the two different types of COPs previously considered in the literature: (i) the simultaneous-offer COP evaluating the contracts of all eligible doctors in each step (e.g., Hatfield and Milgrom (2005)), and (ii) the single-offer COPs evaluating only a single contract at a time (e.g., Hatfield and Kojima (2010)). In the GCOP, an *arbitrary set of eligible doctors* is considered in each step.

Let $AC^0(\succ_d) = AC(\succ_d)$ for each $d \in D$, $UK^0 = \emptyset$, and $X^0 = \emptyset$. The GCOP is defined by the following procedure and finishes in $T \geq 1$ rounds.

Round 1:

[2] The original definition in Aygün and Sönmez (2013) requires that $x \in X \setminus X'$. This formulation is equivalent since the statement immediately holds if $x \in X'$.

[3] Hirata and Kasuya (2014) and Hatfield *et al.* (2021) assume the IRC condition.

– Choose an arbitrary non-empty set of doctors $D^1 \subseteq D$ and identify the most preferred contract, $x_d^1 \in AC^0(\succ_d)$, of each $d \in D^1$ according to $\succ_d$.

Update:

- Let $\tilde{X}^1$ be the set containing x_d^1 for each $d \in D^1$, and let $X^1 = \tilde{X}^1 \cup X^0$.[4]
- For each doctor $d \in D^1$, make x_d^1 unavailable in later rounds;

$$AC^1(\succ_d) = \left\{ \begin{array}{c} AC^0(\succ_d) \setminus \{x_d^1\} \\ AC^0(\succ_d) \end{array} \right\} \text{ if } d \left\{ \begin{array}{c} \in \\ \notin \end{array} \right\} D^1.$$

For each $t \geq 1$, we define X^t and $AC^t(\succ_d)$ recursively. For each $t \geq 1$, X^t is the set of contracts to be considered in round t. For each doctor $d \in D$ and each $t \geq 1$, we call $AC^t(\succ_d)$ the set of *fresh* contracts and $AC(\succ_d) \setminus AC^t(\succ_d)$ the set of *offered* contracts.

– For each hospital $h \in H$, the contracts in $C_h(X^1)$ are reserved.

Update:

- Let $U^1 \coloneqq \{d \in D \mid AC^1(\succ_d) = \emptyset\}$ be the set of doctors with no fresh contracts for later rounds.
- Let $K^1 \coloneqq \bigcup_{h \in H} d(C_h(X^1))$ be the set of doctors with offered contracts reserved by hospitals. Their fresh contracts will not be considered in the next round.
- Let $UK^1 \coloneqq U^1 \cup K^1$ be the set doctors whose fresh contracts may not be reviewed in the next round.

For each doctor $d \in D \setminus UK^1$, no hospital reserves her contract and she has fresh contracts available for later rounds. In other words, $D \setminus UK^1$ is the set of doctors who can be included in D^2. If $UK^1 = D$, the process is complete and stops at $T = 1$. Otherwise, the process moves to the next round.

Round $t \geq 2$:

– Choose an arbitrary non-empty set of doctors $D^t \subseteq D \setminus UK^{t-1}$ and identify the most preferred contract, $x_d^t \in AC^{t-1}(\succ_d)$, of each $d \in D^t$ according to $\succ_d$.

Update:

- Let $\tilde{X}^t$ be the set containing x_d^t for each $d \in D^t$, and let $X^t \coloneqq \tilde{X}^t \cup X^{t-1}$.

[4]Note that since $X^0 = \emptyset$, $\tilde{X}^1 \cup X^0 = \tilde{X}^1$. We use this expression to be consistent across different rounds.

- For each doctor $d \in D^t$, make x_d^t unavailable in later rounds;

$$AC^t(\succ_d) = \left\{ \begin{array}{c} AC^{t-1}(\succ_d) \setminus \{x_d^t\} \\ AC^{t-1}(\succ_d) \end{array} \right\} \text{ if } d = \left\{ \begin{array}{c} \in \\ \notin \end{array} \right\} D^t.$$

– For each hospital $h \in H$, the contracts in $C_h(X^t)$ are reserved.

Update:

- Let $U^t \coloneqq \{d \in D \mid AC^t(\succ_d) = \emptyset\}$,
- let $K^t \coloneqq \bigcup_{h \in H} d(C_h(X^t))$, and
- let $UK^t \coloneqq U^t \cup K^t$.

If $UK^t = D$, the process is complete and stops at $T = t$. Otherwise, the process moves to the next round.

If $D^t = D \setminus UK^{t-1}$ for each $t \in \{1, \ldots, T\}$, the process corresponds to the simultaneous-offer COP. If $|D^t| = 1$ for each $t \in \{1, \ldots, T\}$, the process corresponds to a single-offer COP. We call the resulting set of contracts reserved by hospitals, $\bigcup_{h \in H} C_h(X^T)$, an *outcome*. We say that two GCOPs are *outcome-equivalent* if they have the same outcome.

4 Generalized Order Independence

Hirata and Kasuya (2014) compared single-offer COPs and the simultaneous-offer COP and showed that (i) any two single-offer COPs are outcome-equivalent and (ii) any single-offer COP and the simultaneous-offer COP are outcome-equivalent, assuming that the choice functions of all hospitals satisfy the HK condition described below. This means that the order in which contracts are considered does not affect the outcome of single-offer COPs, and that all single-offer COPs induce the same outcome as the simultaneous-offer COP. Hirata and Kasuya (2014) showed that the combination of the IRC and bilateral substitutability conditions implies the following condition.[5]

Definition 2. *Hospital h's choice function C_h satisfies the* ***Hirata-Kasuya*** *(HK) condition if for any $d, d' \in D$ with $d \neq d'$, any $x \in X_d$, and any $X' \subseteq X$ with $d, d' \notin d(C_h(X'))$, $d' \notin d(C_h(X' \cup \{x\}))$.*

The condition says that given X', if hospital h chooses no contracts involving two doctors, d and d', making a contract with doctor d available to hospital h does not make it choose a contract with doctor d'.

[5] The condition is stated as a lemma in Hirata and Kasuya (2014). The bilateral substitutability condition is introduced by Hatfield and Kojima (2010). Hospital h's choice function satisfies the bilateral substitutability condition if there do not exist a pair of contracts $x, y \in X$ and a set of contracts $X' \subseteq X$ such that (i) $d(x), d(y) \notin d(X')$, (ii) $x \notin C_h(X' \cup \{x\})$, and (iii) $x \in C_h(X' \cup \{x, y\})$.

We generalize the results in Hirata and Kasuya (2014) by showing that all GCOPs are outcome-equivalent, assuming that choice functions satisfy the HK condition. This means that the set of contracts considered in each round of a GCOP has no impact on its outcome and that all GCOPs induce the same outcome as the simultaneous-offer COP. To establish this, we show that for any GCOP, there always exists a single-offer COP which replicates it. Combined with the outcome equivalence of single-offer COPs, the generalized order independence result immediately follows.

Theorem 1. *Given $\succ_D$ and C_H, if C_h satisfies the HK condition for each hospital $h \in H$, then all GCOPs are outcome-equivalent.*

The following example demonstrates the order dependence of GCOPs when hospitals' choice functions do not satisfy the HK condition.

Example 1. We have two doctors $D = \{d_1, d_2\}$ and one hospital, $H = \{h\}$. The set of contracts is $X = \{x_1, x_1', x_2\}$ where $X_1 \cap X_h = \{x_1, x_1'\}$ and $X_2 \cap X_h = \{x_2\}$. Both the doctors' preferences and the hospital's choice function are given below.

$$x_1 \succ_{d_1} x_1' \succ_{d_1} x^\emptyset \quad \text{and} \quad x_2 \succ_{d_2} x^\emptyset$$

$$\begin{array}{rcc}
C_h(\{x_1\}) = \emptyset & C_h(\{x_1, x_1'\}) = \{x_1\} & C_h(X) = \emptyset \\
C_h(\{x_1'\}) = \{x_1'\} & C_h(\{x_1, x_2\}) = \{x_1\} & \\
C_h(\{x_2\}) = \{x_2\} & C_h(\{x_1', x_2\}) = \{x_2\} &
\end{array}$$

All possible GCOPs are visualized below. They show that the matching outcomes crucially depend on which sets of contracts are offered.

$$\begin{tikzcd}
 & C_h(\{x_1, x_1'\}) = \{x_1\} \arrow[r] & C_h(\{x_1, x_1', x_2\}) = \emptyset \\
C_h(\{x_1\}) = \emptyset \arrow[ur] \arrow[dr] & & \\
 & C_h(\{x_1, x_2\}) = \{x_1\} & \\
C_h(\{x_2\}) = \{x_2\} \arrow[r] & C_h(\{x_1, x_2\}) = \{x_1\} & \\
C_h(\{x_1, x_2\}) = \{x_1\} & &
\end{tikzcd}$$

Example 1 is a modified version of Example 1 in Aygün and Sönmez (2012). In this example, the hospital's choice function satisfies the bilateral substitutability condition, but not the IRC condition. In the example of Hirata and Kasuya (2014) which shows (i) the order dependence of the single-offer COP and (ii) the difference beween the single-offer and the simultaneous-offer COPs, the hospitals' choice functions satisfy the IRC condition, but not the bilateral substitutability condition. The two examples show that neither the IRC condition nor the bilateral substitutability condition is sufficient for ensuring order independence. However, the two conditions jointly ensure order independence as they imply the HK condition, which guarantees order independence by Theorem 1.

One might be interested in whether it is possible to find an example where the HK condition is violated and all singe-offer COPs are outcome-equivalent, yet there is a GCOP that produces a different outcome. We have been unable to find such an example and conjecture that for any example in which the HK condition is violated and some non-single-offer GCOPs are not outcome-equivalent, there will exist corresponding single-offer GCOPs that are not outcome-equivalent. This would not be surprising as we have shown in the proof of Theorem 1 that for any GCOP there exists a corresponding single-offer GCOP that produces the same outcome while, in a weak sense, evaluating contracts in the same order. If this conjecture is true, the underlying forces that induce order dependence when HK is violated are the same for non-single-offer GCOPs as for single-offer COPs.

5 Relation with Hatfield *et al.* (2021)

The order independence results of Hirata and Kasuya (2014) rely on their Theorem 1, which shows that the single-offer process is outcome-equivalent under the HK condition, also shown as Lemma 3 in Appendix A. Hatfield *et al.* (2021) also provided a condition (discussed in Appendix B) under which the outcome-equivalence of single-offer processes (Proposition 3 in Hatfield *et al.* (2021)) is established.[6] Hatfield *et al.* (2021) (footnote 43) state that this result generalizes that of Hirata and Kasuya (2014), since their condition is weaker than the bilateral substitutability condition that Hirata and Kasuya (2014)'s outcome-equivalence result relies on.

One natural question is the following: can the outcome-equivalence result in Theorem 1 be established under the condition from Hatfield *et al.* (2021)?

Two points. First, Hatfield *et al.* (2021) focuses only on the sequences of contracts which could arise in single-offer COPs, while the HK condition imposes restrictions on arbitrary sets of contracts. GCOPs allow multiple contracts to be evaluated in each step and the condition in Hatfield *et al.* (2021) (unlike the HK condition) imposes no structure in such cases. Second, as we observed above, the results in Hirata and Kasuya (2014) rather rely on their Lemma (Definition 2), which is implied by the bilateral substitutability condition (in addition to the IRC condition, which is assumed throughout). The relationship between the condition in Hirata and Kasuya (2014) (Definition 2) and that in Hatfield *et al.* (2021) is still unclear.

To address the question above, we first provide a condition, which we term the Hatfield-Kominers-Westkamp (HKW) condition. This is a natural extension of the original condition from

[6] Hatfield *et al.* (2021) also assume the IRC condition.

Hatfield *et al.* (2021) called *observable substitutability across doctors*.[7] As mentioned above, the focus of Hatfield *et al.* (2021) is on the sequences of contracts which could arise under the single-offer COP while the HK condition imposes restrictions on arbitrary sets of contracts. When a GCOP evaluates a single contract in each step, the generated sequence of contracts *represents a single-offer COP* (see the proof of Theorem 1 for details). For any sequence of contracts that represents a single-offer COP, the HKW condition coincides with the condition of observable substitutability across doctors. In other words, the difference stems only from sequences of contracts that cannot be realized under single-offer COPs. The HKW condition extends observable substitutability across doctors to also impose structure on sequences of contracts that could arise in GCOPs but not in single-offer COPs.[8] We show below that when the condition is extended in this way, it becomes equivalent to the HK condition. This implies that the HK and the HKW conditions can be used interchangeably in the current framework.

For each hospital h, let $R_h(X') \coloneqq X'_h \setminus C_h(X')$ be the contracts in X'_h rejected by h. We first provide the Hatfield-Kominers-Westkamp (HKW) condition.

Definition 3. *Hospital h's choice function C_h satisfies the* ***Hatfield-Kominers-Westkamp*** *(HKW) condition if for any $X' \subseteq X$, any $d \notin d(C_h(X'))$ and any $x \in (X_d \cap X_h) \setminus X'_d$, $x' \in R_h(X') \setminus R_h(X' \cup \{x\})$ implies $d(x') \in d(C_h(X'))$.*

The HKW condition can be read as follows. Given X', suppose that a contract x with doctor $d \notin C_h(X')$ is made available to hospital h. If there is a contract x' which is rejected at X' and then accepted at $X' \cup \{x\}$ by hospital h, there must exist another contract with doctor $d(x')$, $\tilde{x}$, which is accepted at X' and rejected at $X' \cup \{x\}$. In other words, the availability of x to hospital h makes it switch from $\tilde{x}$ to x', both of which belong to the same doctor $d(x')$.

Proposition 1. *Hospital h's choice function C_h satisfies the HK condition if and only if it satisfies the HKW condition.*

The proof can be found in Appendix A.3. As we will show in what follows, the result above immediately implies that GCOPs produce stable allocations.

Definition 4 (Hatfield and Milgrom (2005), Hatfield *et al.* (2021))**.** *An allocation $X' \subseteq X$ is* ***stable*** *if*

[7] Appendix B describes the framework as well as the condition in Hatfield *et al.* (2021).

[8] Observable substitutability across doctors only imposes structure for *observable* offer processes for each h. An observable offer process for h is a finite sequence of distinct contracts involving h with the property that the doctor involved in the tth contract is not involved in any of the contracts reserved by h in step $t-1$ (note that these "steps" do not necessarily coincide with the steps in the algorithm described above as they skip any steps in which h does not receive any new offers). While GCOPs do not necessarily generate sequences of single contracts for each h, any sequence of *sets* of contracts involving some hospital h generated by a GCOP is guaranteed to be observable in the sense that none of the fresh contracts reserved by h in step t can involve doctors with contracts reserved by h in step $t-1$. This extended notion of observability reduces to observability as defined by Hatfield *et al.* (2021) when at most one contract per hospital is evaluated in each step. Since the focus of this paper is on sequences of sets of contracts generated by GCOPs and since any such sequences satisfy this extended observability condition, there is no need to impose any explicit requirement of observability in the HKW condition.

1. $\cup_{h\in H} C_h(X') = X'$ *and* $x'_d \succeq_d x^{\emptyset}$ *for each* $d \in D$ *where* $X'_d = \{x'_d\}$*, and*

2. *There does not exist a non-empty set of contracts* $X'' \subseteq X \backslash X'$ *such that* $X'' \subseteq \cup_{h\in H} C_h(X' \cup X'')$ *and* $x''_d \succ_d x'_d$ *for each* $d \in d(X'')$ *where* $X'_d = \{x'_d\}$ *and* $X''_d = \{x''_d\}$*.*

In other words, an allocation is stable if it (i) contains no contract that is deemed unacceptable by the doctor or the hospital it involves and (ii) there is no doctor who would prefer to replace its contract in the allocation with a contract involving some hospital, which in turn would sign this contract if it were available alongside the contracts in the allocation.

Theorem 6 of Hatfield *et al.* (2021) says that observable substitutability across doctors, the HKW condition for the single-offer COP, implies that the single-offer COP is stable. Given that (i) the HKW and HK conditions are equivalent and (ii) all GCOPs are outcome-equivalent under the HK condition, we have the following immediate result as a corollary of Theorem 6 of Hatfield *et al.* (2021).[9]

Corollary 1 (Theorem 6 of Hatfield *et al.* (2021))**.** *If the hospitals' choice functions satisfy the HK condition, any GCOP selects a stable allocation.*

6 Discussion

In this paper, we extended the cumulative offer process to allow for arbitrary subsets of doctors to make proposals at any time, and show that, when hospitals' choice functions satisfy the HK condition, the outcome does not depend on the order and sets of doctors making these proposals at any time. In addition to providing an alternative family of algorithms for the COP, we argue that the model becomes general enough to be able to represent more realistic dynamic processes. In this concluding section, we provide examples of processes that can be modeled as instances of the generalized cumulative offer process. These highlight the flexibility that the model provides to represent many dynamic matching processes.

The key characteristics that a matching process must have to be modeled as a GCOP, in addition to the assumptions about doctors and hospitals preferences we introduced in Section 2, are that (i) doctors can have at most one proposal being held by a hospital at any time, and (ii) the process only ends when there are no doctors waiting to make another proposal or proposals not yet processed by the hospitals. Below we list some examples of scenarios where these characteristics are present.

Doctors and Hospitals working asynchronously Doctors can, each one independently and at any time, make a proposal to a hospital, including the contractual terms that they desire. Similarly, hospitals can, each one independently and at any time, process the pending

[9] Hatfield and Kojima (2010) points out that bilateral substitutability is weaker than substitutability (Hatfield and Milgrom (2005)).

proposals, holding some of these offers, renegotiating some of them, and rejecting others. The process ends when no doctor wants to make some additional proposal and all hospitals processed their pending proposals. This description, with the roles of doctors and hospitals reversed, closely resembles the description of the market for clinical psychologists in Roth and Xing (1997).

Doctors arrive at different times This scenario resembles the dynamics in a job fair. In it, hospitals are active in the market from the beginning. Doctors, however, are not, and arrive individually or in groups. Once there, they make proposals, hospitals process them, holding some and rejecting others. Doctors who are rejected can make new proposals. While this takes place, more doctors can arrive and make their proposals as well. This process goes on until all doctors arrived and there are no doctors wanting to make new offers.[10]

Academic publication process Authors have papers that they would like to have published in journals and have preferences over these journals. Authors can submit their papers at any time. Importantly, a paper cannot be submitted while it is still being reviewed by another journal. Each editor evaluates submissions made since the last time they made these decisions and can issue desk rejections, revision requests, and acceptances.[11] Authors who receive rejections submit their papers to their next most-preferred journal at any time, and acceptances are always final.

Centralized university admissions followed by waitlists in Brazil In Brazil, admissions to public universities is centralized using an on-line dynamic mechanism in which each student applies to one program at a time for a pre-determined number of periods, in a process that, under the assumption that students follow rationalizable strategies, is an instance of GCOP (Bó and Hakimov, 2022). After the dynamic mechanism is used, universities run second, third, fourth and sometimes more rounds of calls among those whose last application was one of their programs. The combination of the two processes—the dynamic centralized mechanism, followed by waitlist rounds—can be modeled as an instance of GCOP. In this model, students who are waitlisted, instead of making their last choice in the centralized dynamic mechanism, "wait" to make their last application in the centralized mechanism in the waitlist rounds.

Statistical reduction of interactions in university admissions Based on historical data, university entrance administrators design the sets of students who are called for making applications in each period such that the total number of times students are called to make a

[10] Notice that a model in which hospitals can also arrive at different times would involve less appealing assumptions: doctors with fresh contracts involving hospitals that have not yet arrived would wait for that to happen before making their next proposal.

[11] Revisions can be modeled as different contracts with the same journal, and every author follows the same order in their "preferences": first submission, first revision, etc.

new proposal is reduced. The identities of the students who are called at each period can be dynamically determined not only on the basis of the historical data, but on the proposals that students make in each period.

It is important, however, to emphasize that we are not claiming that our results say anything about whether we can predict the behavior of strategic agents in these scenarios to be "truthful". That is, it might be that for some of these scenarios some doctors would be better off by making proposals that don't simply follow their preferences, as described in the GCOP. Bó and Hakimov (2022), for example, show that in dynamic processes that resemble DA—an instance of a GCOP—being truthful in their proposals is not always a best response for the agents, despite DA being a strategy-proof mechanism. It is outside of the scope of this paper to evaluate the incentives of doctors and/or hospitals in these dynamic processes. Our results show that when the agents involved make proposals following their preferences over contracts, all of these processes will converge to the same outcome as the COP.

The examples we give above give us a taste of the usefulness of the GCOP as a tool to model real-life dynamic processes, but also as a method to explore possible designs. The last example given above, in which historical data is used to design the set of students who are called to make applications exemplifies this idea and is, to our knowledge, not being used anywhere.[12] The fact that it results in a GCOP indicates that, despite it clearly involving in a complex set of interactions, when students make proposals following their preferences the outcome will still be stable.

Declaration of interest

None.

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[12]The province of Inner Mongolia in China, however, uses a dynamic on-line procedure in its university admissions where historical data is used to *advise* students on their proposals, reducing the number of adjustments that would be necessary otherwise (Inner Mongolia Autonomous Region Education Entrance Examination Center, 2022).

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A Proofs

A.1 Proof of Lemma 1

Consider any $h \in H$ and any $X' \subseteq X$. Let $X^1 = X'_{-h} \setminus \{x_1\}$ for some $x_1 \in X'_{-h}$. Since $C_h(X') \subseteq X'_h$ by definition, $x_1 \in X'_{-h}$ implies that $x_1 \notin C_h(X')$. By IRC, we then have that $C_h(X') = C_h(X'_h \cup X'_{-h}) = C_h(X'_h \cup X^1)$. If $X^1 = \emptyset$, then $C_h(X') = C_h(X'_h \cup X^1) = C_h(X'_h)$. If $X^1 \neq \emptyset$, let $X^2 = X^1 \setminus \{x_2\}$ for some $x_2 \in X^1$. By the same logic as above, IRC requires that $C_h(X') = C_h(X'_h \cup X^1) = C_h(X'_h \cup X^2)$. Repeating this argument yields $C_h(X') = C_h(X'_h \cup X^1) = \cdots = C_h(X'_h \cup X^{|X'_{-h}|}) = C_h(X'_h)$. The final equality follows from the fact that $X^{|X'_{-h}|} = \emptyset$. ♣

A.2 Proof of Theorem 1

Fix $\succ_D$ and C_H. Take a GCOP requiring T rounds of iteration and its corresponding $\tilde{X}^t$, for each $t \in \{1, \ldots, T\}$. Arrange the elements of $\tilde{X}^1 \ldots \tilde{X}^T$ as follows.

$$\underbrace{x^1, \ldots, x^{|X^1|}}_{\tilde{X}^1}, \underbrace{x^{|X^1|+1}, \ldots, x^{|X^2|}}_{\tilde{X}^2}, \ldots, \underbrace{x^{|X^{T-1}|+1}, \ldots, x^{|X^T|}}_{\tilde{X}^T}$$

The internal ordering of the elements within each $\tilde{X}^t$ is arbitrary. Now, consider a different GCOP that evaluates a single contract in each round. To distinguish the original GCOP from this "single-offer" GCOP (sGCOP), we use the notation y, Y and $\tilde{Y}$ for the sGCOP and x, X and $\tilde{X}$ for the original GCOP. Furthermore, we use $\bar{AC}$, $\bar{D}$, $\bar{U}$, $\bar{K}$ and $\bar{UK}$ for the sGCOP and AC, D, U, K and UK for the original GCOP. We call the sequence corresponding to $\tilde{X}^t$ the $\tilde{X}^t$-*sequence*. That is, the $\tilde{X}^t$-sequence is the sequence of contracts given by $x^{|X^{t-1}+1|}, \ldots, x^{|X^t|}$. We define the sGCOP such that $\tilde{Y}^s = x^s$ for all $s \leq |X^T|$.

The proof idea is as follows: First, show that the single-offer GCOP is a GCOP that results in the same outcome as the original GCOP. Second, show that the sequence of contracts considered by the sGCOP represents a single-offer COP as defined by Hirata and Kasuya (2014). This implies that for any GCOP there exists a single-offer GCOP with a corresponding sequence of contracts that both represents a single-offer COP and produces the same outcome. By invoking Hirata and Kasuya's (2014) order independence result, restated as Lemma 3 below, all such single-offer GCOPs can be shown to produce the same outcome. This implies that all GCOPs are outcome-equivalent. To demonstrate that the sGCOP is a GCOP, we must ensure that doctor $d(x^s)$ is available in round s for each round $s \leq |X^T|$ in the sGCOP.

Suppose that the two GCOPs described above have considered the same set of contracts up until the end of the $\tilde{X}^{t-1}$-sequence. That is, the set of contracts that have been considered in the original GCOP after $t-1$ steps equals the set of contracts that have been considered in the sGCOP after $|X^{t-1}|$ rounds; $X^{t-1} = Y^{|X^{t-1}|}$ and $AC^{t-1}(\succ_d) = \bar{AC}^{|X^{t-1}|}(\succ_d)$ for all $d \in D$. Consider round

$|X^{t-1}|+\tau$ in the sGCOP for some $\tau \in \{1,\ldots,|\tilde{X}^t|\}$.

$$\ldots x^{|X^{t-1}|}, \underbrace{x^{|X^{t-1}|+1}, \ldots, x^{|X^{t-1}|+\tau}, \ldots, x^{|X^t|}}_{\tilde{X}^t}, \ldots$$

Let $\tilde{D}^\tau \subseteq D^t$ denote the set of doctors whose contracts are considered in rounds $|X^{t-1}|+1$ through $|X^{t-1}|+\tau$ in the sGCOP, with $|\tilde{D}^\tau| = \tau$ and $\tilde{D}^0 = \emptyset$, and let $\hat{D}^\tau := D^t \setminus \tilde{D}^\tau$.

The following result shows that every doctor in D^t whose contract was not considered in rounds $|X^{t-1}|+1$ through $|X^{t-1}|+\tau$ is still available in the next round of the sGCOP. This means that the remaining contracts in $\tilde{X}^t$ are still available for consideration after some contracts in $\tilde{X}^t$ have already been processed. This result is key to ensuring that a GCOP can process the contracts in $\tilde{X}^t$ in a sequence one by one rather than simultaneously.

Lemma 2. *For any doctor $d \in \hat{D}^\tau$, $d \notin \bar{UK}^{|X^{t-1}|+\tau}$.*

Proof. Consider any contract y for which $d(y) = d \in \hat{D}^\tau$ and note that $d \notin \bar{U}^{|X^{t-1}|+\tau}$, since $\bar{AC}^{|X^{t-1}|}(\succ_d) = \bar{AC}^{|X^{t-1}|+\tau}(\succ_d)$ and $d \notin \bar{U}^{|X^{t-1}|}$ as $d \in D^t$. Furthermore, $d \notin \bigcup_{h\in H} d(C_h(Y^{|X^{t-1}|}))$ for all $d \in D^t$ by construction of the original GCOP. Consider some $\sigma \in \{0,\ldots,\tau\}$ and suppose $d \notin \bigcup_{h\in H} d(C_h(Y^{|X^{t-1}|+\sigma})) = \bar{K}^{|X^{t-1}|+\sigma}$ for all $d \in \hat{D}^\sigma$. This statement holds for $\sigma = 0$ since $\hat{D}^0 = D^t$ and $d \in D^t$. Note that $Y^{|X^{t-1}|+\sigma+1} = Y^{|X^{t-1}|+\sigma} \cup \tilde{Y}^{|X^{t-1}|+\sigma+1}$ and $\tilde{Y}^{|X^{t-1}|+\sigma+1}$ is a singleton. Since all hospitals satisfy the HK condition, $d \notin d(C_h(Y^{|X^{t-1}|+\sigma} \cup \tilde{Y}^{|X^{t-1}|+\sigma+1})) = d(C_h(Y^{|X^{t-1}|+\sigma+1}))$ for all $d \in \hat{D}^{\sigma+1} = \hat{D}^\sigma \setminus \{d(\tilde{Y}^{|X^{t-1}|+\sigma+1})\}$ and all $h \in H$. Thus, $d \notin \bigcup_{h\in H} d(C_h(Y^{|X^{t-1}|+\sigma+1})) = \bar{K}^{|X^{t-1}|+\sigma+1}$ for all $d \in \hat{D}^{\sigma+1}$. That is, if $d \notin \bar{K}^{|X^{t-1}|+\sigma}$ for all $d \in \hat{D}^\sigma$, then $d \notin \bar{K}^{|X^{t-1}|+\sigma+1}$ for all $d \in \hat{D}^{\sigma+1}$ as well. Since $d \notin \bar{K}^{|X^{t-1}|}$ for all $d \in \hat{D}^0$ by construction, it follows that $d \notin \bar{K}^{|X^{t-1}|+\tau}$ for all $d \in \hat{D}^\tau$ by induction. Since $\bar{UK}^{|X^{t-1}|+\tau} = \bar{U}^{|X^{t-1}|+\tau} \cup \bar{K}^{|X^{t-1}|+\tau}$, $d \notin \bar{UK}^{|X^{t-1}|+\tau}$ for all $d \in \hat{D}^\tau$. ♣

Next, we will show that if the two GCOPs have considered the same set of contracts up until the end of the $\tilde{X}^{t-1}$-sequence, it will also have considered the same set of contracts by the end of the $\tilde{X}^t$-sequence. Since the premise holds for the first sequence, the same set of contracts will have been considered by the two GCOPs up until the end of every such sequence, by induction. This observation and Lemma 2 jointly imply that $d(x^s) \in D \setminus \bar{UK}^s$ in every round s of the sGCOP. Thus, the sGCOP is a GCOP. It also implies that $X^T = Y^{|X^T|}$, i.e., the sGCOP and the original GCOP are outcome-equivalent.

Base case: By construction, $X^0 = Y^{|X^0|} = Y^0 = \emptyset$ and $AC^0(\succ_d) = \bar{AC}^{|X^0|}(\succ_d) = \bar{AC}^0(\succ_d) = AC(\succ_d)$ for all $d \in D$.

Induction hypothesis: Assume that there exists some t such that $X^{t-1} = Y^{|X^{t-1}|}$ and $AC^{t-1}(\succ_d) = \bar{AC}^{|X^{t-1}|}(\succ_d)$ for all $d \in D$.

Induction step: We will now demonstrate that $X^t = Y^{|X^t|}$ and $AC^t(\succ_d) = \bar{AC}^{|X^t|}(\succ_d)$ for all

$d \in D$. Consider the $\tilde{X}^t$-sequence.

$$\dots x^{|X^{t-1}|}, \underbrace{x^{|X^{t-1}|+1}, \dots, x^{|X^t|}}_{\tilde{X}^t}, \dots$$

Take $\tau \in \{1, \dots, |\tilde{X}^t|\}$ and let (i) the set of doctors whose contracts are considered in rounds $|X^{t-1}| + 1$ through $|X^{t-1}| + \tau$ in the sGCOP be $\tilde{D}^\tau \subseteq D^t$ with $|\tilde{D}^\tau| = \tau$ and $\tilde{D}^0 = \emptyset$, and (ii) $\hat{D}^\tau := D^t \setminus \tilde{D}^\tau$.

Round $|X^{t-1}|+\tau$: By Lemma 2, any $d' \in \hat{D}^{\tau-1}$ is available in round $|X^{t-1}|+\tau$. Take an arbitrary doctor $d' \in \hat{D}^{\tau-1}$.

D1: Identify doctor d''s most preferred contract in $\bar{AC}^{|X^{t-1}|+\tau-1}(\succ_{d'})$, $y_{d'}$.

Update: Let

- $\tilde{D}^\tau = \tilde{D}^{\tau-1} \cup \{d'\}$,
- $\tilde{Y}^{|X^{t-1}|+\tau} = \{y_{d'}\}$ and $Y^{|X^{t-1}|+\tau} = \tilde{Y}^{|X^{t-1}|+\tau} \cup Y^{|X^{t-1}|+\tau-1}$, and
- $\bar{AC}^{|X^{t-1}|+\tau}(\succ_{d'}) = \bar{AC}^{|X^{t-1}|+\tau-1}(\succ_{d'}) \setminus \{y_{d'}\}$ and $\bar{AC}^{|X^{t-1}|+\tau}(\succ_d) = \bar{AC}^{|X^{t-1}|+\tau-1}(\succ_d)$ for each $d \neq d'$. Note that $\bar{AC}^{|X^{t-1}|+\tau}(\succ_{d'}) = AC^t(\succ_{d'})$.

H1: Let $h(y_{d'}) = h'$. Hospital h' reserves $y_{d'}$ if $y_{d'} \in C_{h'}(Y^{|X^{t-1}|+\tau})$.

Note that $\bar{AC}^{|X^0|}(\succ_d) = AC(\succ_d)$ for all $d \in D$ and that, by construction, any $x \in \bar{AC}^{|X^0|}(\succ_d) \setminus \bar{AC}^{|X^t|+\tau}(\succ_d)$ has been considered in rounds 1 through $|X^t| + \tau$.

Update: Let

- $\bar{U}^{|X^{t-1}|+\tau} = \{d \in D \mid \bar{AC}^{|X^{t-1}|+\tau}(\succ_d) = \emptyset\}$,
- $\bar{K}^{|X^{t-1}|+\tau} = \bigcup_{h \in H} d(C_h(Y^{|X^{t-1}|+\tau}))$, and
- $\bar{UK}^{|X^{t-1}|+\tau} = \bar{U}^{|X^{t-1}|+\tau} \cup \bar{K}^{|X^{t-1}|+\tau}$.

The set of doctors that can be considered in the next round is given by $D \setminus \bar{UK}^{|X^{t-1}|+\tau}$, where $\hat{D}^\tau \subseteq D \setminus \bar{UK}^{|X^{t-1}|+\tau}$ by Lemma 2. That is, at round $|X^{t-1}|+\tau$ for any $\tau \in \{1, \dots, |\tilde{X}^t|\}$, all of the doctors in D^t whose contracts in $\tilde{X}^t$ were not considered in rounds $|X^{t-1}| + 1$ through $|X^{t-1}| + \tau$ are still available for consideration in round $|X^{t-1}| + \tau + 1$. In each round from $|X^{t-1}| + 1$ to $|X^{t-1}| + \tau$ a new unique contract in $\tilde{X}^t$ is considered. By varying the values of t and τ, the process above describes any round in the sGCOP. Letting $\tau = |\tilde{X}^t|$ implies that all contracts in $\tilde{X}^t$ have been considered in round $|X^{t-1}| + |\tilde{X}^t| = |X^t|$ of the sGCOP. Thus, $Y^{|X^t|} = Y^{|X^{t-1}|} \cup \tilde{X}^t$. Since $X^t = X^{t-1} \cup \tilde{X}^t$, the induction hypothesis then implies that $AC^t(\succ_d) = \bar{AC}^{|X^t|}(\succ_d)$ for all $d \in D$.

Furthermore, since $X^{t-1} = Y^{|X^{t-1}|}$, it follows that $X^t = Y^{|X^t|}$ as $X^t = X^{t-1} \cup \tilde{X}^t = Y^{|X^{t-1}|} \cup \tilde{X}^t = Y^{|X^t|}$. This concludes the induction step.

By induction, we have shown that, for all $t \leq T$, $X^t = Y^{|X^t|}$ and $AC^t(\succ_d) = \bar{AC}^{|X^t|}(\succ_d)$ for all $d \in D$. Thus, the original GCOP terminates in round T and the sGCOP terminates in round $|X^T|$, where $X^T = Y^{|X^T|}$. Consequently, $C_h(X^T) = C_h(Y^{|X^T|})$ for each $h \in H$. This implies that $\bigcup_{h\in H} C_h(X^T) = \bigcup_{h\in H} C_h(Y^{|X^T|})$. In other words, the original GCOP and the sGCOP are outcome-equivalent.

By first showing that the sequence of contracts $x^1, \ldots, x^{|X^T|}$ considered in the sGCOP represents a single-offer COP, we can apply the order independence result of Hirata and Kasuya (2014).

Definition 5 (Hirata and Kasuya (2014, Definition 4)). *Given $\succ_D$ and C_H, a finite sequence of contracts $(x^t)_{t=1}^T$* ***represents a single-offer COP*** *if the following conditions are satisfied:*

(1) For each $t \in \{1, \ldots, T\}$ and $h \in H$, $d(x^t) \notin d(C_h(\{x^1, \ldots, x^{t-1}\}))$.[13]

(2) For each $t \in \{1, \ldots, T\}$ and $x \in X$, if $d(x^t) = d(x)$ and $x \succ_{d(x)} x^t$, there exists $\tau < t$ such that $x = x^\tau$.

(3) For each $d \in D$, either (i) $d \in d(C_h(\{x^1, \ldots, x^T\}))$ for some $h \in H$ or (ii) $AC(\succ_d) \subseteq \{x^1, \ldots, x^T\}$.[14]

Note that, by construction, no doctors involved in the contracts in $\tilde{X}^t$ are reserved in round $t-1$ of any GCOP. That is, $d(x) \notin C_h(X^{t-1})$ for each $x \in \tilde{X}^t$, each $h \in H$, and each round $t \leq T$ of any GCOP, including the sGCOP. Since $\tilde{Y}^s = x^s$ in each round $s \leq |X^T|$ of the sGCOP, the sequence of contracts $x^1, \ldots, x^{|X^T|}$ considered in the sGCOP satisfies condition (1) in Definition 5.

Next, consider a doctor $d \in D$ and a round s of the sGCOP. If there exists some $x \in X$ such that $d(x^s) = d(x)$ and $x \succ_{d(x)} x^s$, then $x \in \bar{AC}(\succ_d) \setminus \bar{AC}^s(\succ_d)$ since x^s is d's most preferred contract in $\bar{AC}^s(\succ_d)$. By construction, $x \in \bar{AC}(\succ_d) \setminus \bar{AC}^s(\succ_d)$ implies that $x \in \tilde{Y}^\tau$ for some round $\tau < s$. Since $\tilde{Y}^s = x^s$ in each round s, the sequence of contracts considered in the sGCOP satisfies condition (2) in Definition 5 as well.

Since $X^T = Y^{|X^T|}$, the GCOP and the sGCOP have considered the same contracts in rounds T and $|X^T|$ of the GCOP and sGCOP, respectively. The GCOP terminates in round T where $UK^T = D$. This means that $d \in U^T \cup K^T$ for all $d \in D$.

(a) If $d \in U^T$, then $AC^T(\succ_d) = \bar{AC}^{|X^T|}(\succ_d) = \emptyset$. Note that $AC^0(\succ_d) = \bar{AC}^{|X^0|}(\succ_d) = AC(\succ_d)$ for each $d \in D$ and that the set of contracts involving d that have been considered in rounds 1 through $|X^t| + \tau$ in the sGCOP is given by $AC(\succ_d) \setminus \bar{AC}^{|X^t|+\tau}(\succ_d)$. $\bar{AC}^{|X^T|}(\succ_d) = \emptyset$

[13]This corresponds to the notion of observability in Hatfield *et al.* (2021).

[14]Note that (i) and (ii) are not mutually exclusive. There may exist doctor d whose least-preferred contract in $AC(\succ_d)$ is accepted in round T.

implies that $AC(\succ_d) \setminus \bar{AC}^{|X^T|}(\succ_d) = \bar{AC}(\succ_d)$. In other words, all contracts in $AC(\succ_d)$ have been considered in round $|X^T|$ of the sGCOP. This means that $AC(\succ_d) \subseteq Y^{|X^T|}$.

(b) If $d \in K^T$, then $d \in \bigcup_{h\in H} d(C_h(X^T)) = \bigcup_{h\in H} d(C_h(Y^{|X^T|}))$.

Both cases (a) and (b) jointly imply that the sequence of contracts considered in the sGCOP satisfies condition (3) in Definition 5. Since conditions (1), (2) and (3) in Definition 5 are satisfied, the sequence of contracts considered in the sGCOP represents a single-offer COP.

Hirata and Kasuya (2014) have demonstrated that, given some $\succ_D$ and C_H, all single-offer COPs are outcome-equivalent.

Lemma 3. *(Hirata and Kasuya (2014, Theorem 1)) Suppose that two sequences of contracts represent some single-offer COPs at $\succ_D$ and C_H. If every C_h satisfies the HK and IRC conditions, then they induce the same set of contracts as their outcome.*[15]

Given any $\succ_D$ and C_H, we have shown that for any GCOP, there exists some outcome-equivalent GCOP that evaluates contracts one by one in a sequence that represents a single-offer COP. This implies that all GCOPs are outcome-equivalent, since all single-offer COPs are outcome-equivalent by Lemma 3. ♣

A.3 Proof of Proposition 1

[HK implies HKW] Suppose that C_h violates the HKW condition. This implies that there exist (i) $X' \subseteq X$, (ii) $d \notin d(C_h(X'))$, (iii) $x \in (X_d \cap X_h) \setminus X'_d$ and (iv) $x' \in R_h(X') \setminus R_h(X' \cup \{x\})$ such that $d(x') \notin d(C_h(X'))$.[16] Note that $x' \in R_h(X') \setminus R_h(X' \cup \{x\})$ implies (i) $x' \notin C_h(X')$, and (ii) $x' \notin R_h(X' \cup \{x\})$ and thus $x' \in C_h(X' \cup \{x\})$.

- Suppose $d = d(x')$. Since C_h chooses an allocation and allocations cannot contain more than one contract per doctor, $x' \in C_h(X' \cup \{x\})$ implies $x \notin C_h(X' \cup \{x\})$. The combination of $x' \notin C_h(X')$ and $x' \in C_h(X' \cup \{x\})$ implies $C_h(X') \neq C_h(X' \cup \{x\})$. However, since $x \notin C_h(X' \cup \{x\})$, the IRC condition requires that $C_h(X') = C_h(X' \cup \{x\})$. This is a contradiction. Thus, $d \neq d(x')$.

- Suppose instead that $d \neq d(x')$. We have $X' \subseteq X$ with $d, d(x') \notin d(C_h(X'))$. Since $x' \in C_h(X' \cup \{x\})$, it follows that $d(x') \in d(C_h(X' \cup \{x\}))$. This violates the HK condition.

Thus, a violation of the HKW condition implies that the HK condition is violated, or equivalently, the HK condition implies the HKW condition.

[15] Theorem 1 in Hirata and Kasuya (2014) uses the bilateral substitutability condition rather than the HK condition. However, bilateral substitutability is only used to ensure that the HK condition is satisfied in their proof. Their Theorem 1 can therefore be rephrased as in Lemma 3.

[16] Since $x \notin X'_d$ and $x' \in X'$, $x \neq x'$ independent of whether $d(x) = d(x')$ or $d(x) \neq d(x')$.

[HKW implies HK] Suppose that C_h violates the HK condition. Then there exist (i) $d, d' \in D$ with $d \neq d'$, (ii) $x \in X_d$, and (iii) $X' \subseteq X$ with $d, d' \notin d(C_h(X'))$ such that $d' \in d(C_h(X' \cup \{x\}))$. Note that Lemma 1 immediately implies that $x \in X_h$.

- If $x \in X'_d$, this leads to a contradiction since it implies $X' \cup \{x\} = X'$, while $d' \in d(C_h(X' \cup \{x\})) \setminus d(C_h(X'))$. Thus, $x \in (X_d \cap X_h) \setminus X'_d$.
- Since $d' \notin d(C_h(X'))$ and $d' \in d(C_h(X' \cup \{x\}))$, there must exist some $x' \in X'_{d'}$ such that $x' \in R_h(X') \setminus R_h(X' \cup \{x\})$.[17] Then the HKW condition is violated, since $d' \notin d(C_h(X'))$.

Thus, a violation of the HK condition implies that the HKW condition is violated, or equivalently, the HKW condition implies the HK condition. ♣

B Observable Substitutability across Doctors

We first provide the framework of Hatfield *et al.* (2021). Hatfield *et al.* (2021) define an *offer process for h* as a finite sequence of distinct contracts $(x^1, \dots, x^m)$, where $x^\tau \in X_h$ for all $\tau \in \{1, \dots, m\}$. An offer process for h, $(x^1, \dots, x^m)$, is *observable* if $d(x^\tau) \notin d(C_h(\{x^1, \dots, x^{\tau-1}\}))$ for all $\tau \in \{1, \dots, m\}$. In other words, the doctor involved in the τth contract is not involved in any of the contracts chosen by h when the first $\tau - 1$ contracts are considered.

Let $\vdash$ represent a strict ordering of the elements of X determining which contract is considered in each round. Given $\succ_D$ and $\vdash$, the single-offer COP in Hatfield *et al.* (2021) is defined by the following procedure: First, let $A^0 := \emptyset$ be the set of contracts available to hospitals.

Round $t \geq 1$: Consider the following set:

$$U^t := \left\{ x \in X \setminus A^{t-1} \;\middle|\; \begin{array}{l} d(x) \notin d(C_h(A^{t-1})) \text{ for all } h \in H, \text{ and} \\ \nexists x' \in (X_{d(x)} \setminus A^{t-1}) \cup \{x^\emptyset\} \text{ such that } x' \succ_{d(x)} x \end{array} \right\}$$

If $U^t = \emptyset$, the process is complete and stops. Otherwise, let $\tilde{x}$ be the highest-ranked element of U^t according to $\vdash$, and let $A^t := A^{t-1} \cup \{\tilde{x}\}$. Identify $C_h(A^t)$ for all $h \in H$ and move to the next round.

Note that only one contract is considered in each round and that the first condition implies that the process is observable. Similarly, the resulting offer process for any single-offer COP in our framework is observable since the fresh contracts of doctors with reserved contracts in round $t-1$ are not considered in round t. Hatfield *et al.* (2021) show that under the following condition, any two single-offer COPs lead to the same outcome.

[17] Note that $x' \neq x$ since $d(x') = d' \neq d = d(x)$.

Definition 6 (Hatfield, Kominers, and Westkamp (2021, Definition 8))**.** *Hospital h's choice function, C_h, is **observably substitutable across doctors** if, for any observable offer process $(x^1, \ldots, x^m)$ for h, $x \in R_h(\{x^1, \ldots, x^{m-1}\}) \backslash R_h(\{x^1, \ldots, x^m\})$ implies $d(x) \in d(C_h(\{x^1, \ldots, x^{m-1}\}))$.*

That is, if hospital h rejects x at $\{x^1, \ldots, x^{m-1}\}$ and then reserves it at $\{x^1, \ldots, x^m\}$, i.e., when the mth contract is added to the offer process for h, there exists another contract, $\tilde{x} \neq x$, involving the same doctor, $d(x) = d(\tilde{x})$, which is reserved by hospital h at $\{x^1, \ldots, x^{m-1}\}$. In other words, the availability of x^m makes hospital h switch from $\tilde{x}$ to x, both of which involve the same doctor. Note (i) that $\tilde{x} \notin C_h(\{x^1, \ldots x^m\})$ since $x \in C_h(\{x^1, \ldots x^m\})$ and (ii) that $d(x^m) \neq d(x)$ since the offer process is observable.

While observable substitutability across doctors only imposes structure on observable offer processes in Hatfield *et al.* (2021), the HKW condition imposes an analogous requirement for any subset of contracts. As such, the HKW condition is stronger than observable substitutability across doctors when considering GCOPs. However, the conditions are equivalent when focusing on single-offer COPs.

The following is the outcome-equivalence result for single-offer COPs in Hatfield *et al.* (2021).

Proposition 2 (Hatfield, Kominers, and Westkamp (2021, Proposition 3))**.** *If C_h is observably substitutable across doctors for each $h \in H$, for any $\succ_D$ and any two orderings $\vdash$ and $\vdash'$, the outcome with $\vdash$ is identical to the outcome with $\vdash'$.*